MATHEMATICS FOR MECHANICAL ENGINEERS

Problems and Solutions

S. H. OMRAN • M. T. CHAUHAN
H. M. HUSSEN • N. G. NACY • L. J. HABEEB

Mathematics for Mechanical Engineers

Mathematics for Mechanical Engineers

Problems and Solutions

S. H. Omran

M. T. Chaichan

H. M. Hussen, PhD

N. G. Nacy, PhD

L. J. Habeeb, PhD

Mercury Learning and Information
Dulles, Virginia
Boston, Massachusetts
New Delhi

Publisher: David Pallai
Mercury Learning and Information
22841 Quicksilver Drive
Dulles, VA 20166
info@merclearning.com
www.merclearning.com
800-232-0223

S. H. Omran et al. *Mathematics for Mechanical Engineers: Problems and Solutions*.
ISBN: 978-1-68392-790-7

Library of Congress Control Number: 2021945532

212223 321 This book is printed on acid-free paper in the United States of America.

CONTENTS

CHAPTER 1

THE RATE OF CHANGE OF A FUNCTION

PROBLEMS

PROBLEM 1.1

The steel in railroad tracks expand when heated. For the track temperature encountered in normal outdoor use, the length *S* of a piece of track is related to its temperature *t* by a linear equation. An experiment with a piece of track gave the following measurements:

$$\mathbf{t_1 = 65°F,\ s_1 = 35\,ft}$$

$$\mathbf{t_2 = 135°F,\ s_2 = 35.16\,ft}$$

Write a linear equation for the relation between *s* and *t*.

Solution: $p_1(65, 35), p_2(135, 35.16)$

$$\frac{s - s_1}{t - t_1} = \frac{s_2 - s_1}{t_2 - t_1} \Rightarrow \frac{s - 35}{t - 65} = \frac{35.16 - 35}{135 - 65}$$

$$\Rightarrow \quad \frac{s - 35}{t - 65} = \frac{0.16}{70} = 0.0023$$

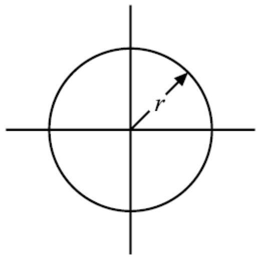
$$\therefore \quad \frac{s - 35}{t - 65} = 0.0023 \Rightarrow 0.0023t - 0.1495 = s - 35$$

$$s = 0.0023t + 34.85$$

PROBLEM 1.2

Three of the following four points lie on a circle whose center is at the origin. What are they and what is the radius of the circle?
$A(-1, 7)$, $B(5, -5)$, $C(-7, 5)$, and $D(7, -1)$

Solution:



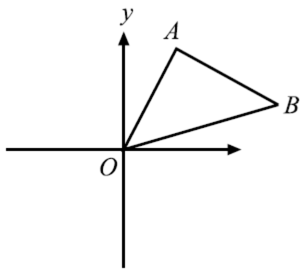
$$r^2 = x^2 + y^2$$
$$rA^2 = (-1)^2 + (7)^2 = 50$$
$$rB^2 = (5)^2 + (-5)^2 = 50$$
$$rC^2 = (-7)^2 + (75)^2 = 74$$
$$rD^2 = (7)^2 + (-1)^2 = 50$$

$\therefore$ A and B and D lie in the circle and radius $r = \sqrt{50}$.

PROBLEM 1.3

A and B are the points (3, 4) and (7, 1), respectively. Use Pythagorian theorem to prove that OA is perpendicular to AB. Calculate the slopes of OA and AB, and find their product.

Solution: The points are $A(3, 4)$, $B(7, 1)$, and $O(0, 0)$.



Now, $$OB^2 = OA^2 + AB^2$$
$$OB = \sqrt{(7-0)^2 + (1-0)^2} = \sqrt{50}$$
$$AB = \sqrt{(7-3)^2 + (1-4)^2} = 5$$
$$\text{OA} = \sqrt{(3-0)^2 + (4-0)^2} = 5$$
$$OB^2 = OA^2 + AB^2$$
$$\Rightarrow \quad (\sqrt{50})^2 = (5)^2 + (5)^2$$
$$= 25 + 25$$

$\therefore \text{OA} \perp \text{AB}$

$$\text{Slope of } AB(mAB) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-4}{7-3} = -\frac{3}{4}$$

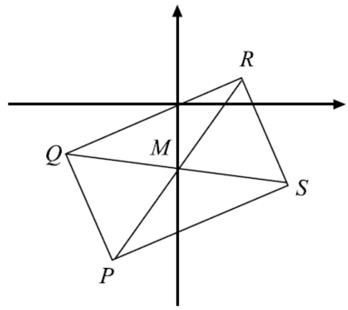
$$\text{Slope of } OA(mOA) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-0}{3-0} = \frac{4}{3}$$

$$\therefore \text{ Slope of } AB \times \text{Slope of } OA = \frac{-3}{4} \times \frac{4}{3} = -1$$

PROBLEM 1.4

$P(-2,-4)$, $Q(-5,-2)$, $R(2,1)$, and S are the vertices of a parallelogram. Find the coordinates of M, and the point of intersection of the diagonals and of S.

Solution: $P(-2,-4), Q(-5,-2), R(2,1), S$



$$\text{Mid-point } PR = \frac{2-2}{2} = 0,$$

$$\frac{1-4}{2} = \frac{-3}{2}$$

$\therefore$ Coordinates of M are $\left(0, \frac{-3}{2}\right)$

$$\text{Mid-point } QS = \frac{x-5}{2} = 0, \frac{y-2}{2} = -\frac{3}{2}$$

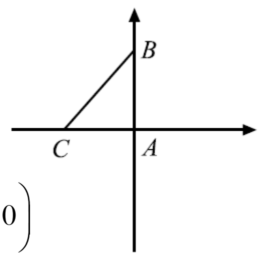
$\therefore\ S(5,-1)$

Ans. $M(0,-3/2), S(5,-1)$

PROBLEM 1.5

Calculate the area of the triangle formed by the line $3x-7y+4=0$ and the axes.

Solution:



$$3x-7y+4=0$$

at $\quad x=0 \Rightarrow 7y=4 \quad \therefore y=\frac{4}{7}\left(0,\frac{4}{7}\right)$

at $\quad y=0 \Rightarrow 3x=-4 \quad \therefore x=\frac{-4}{3}\left(\frac{-4}{3},0\right)$

$\therefore \quad A(0,0), B\left(0,\frac{4}{7}\right), C\left(\frac{-4}{3},0\right)$ are the vertices of the triangle ABC.

$$AB = \sqrt{(0-0)^2 + \left(\frac{4}{7}-0\right)^2} = \frac{4}{7}$$

$$AC = \sqrt{\left(\frac{-4}{3}-0\right)^2 (0-0)^2} = \frac{4}{3}$$

The area of the triangle $\frac{1}{2}AB \times AC$

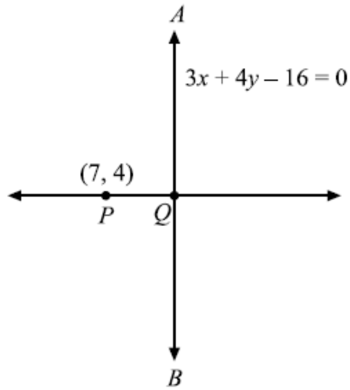
$$= \frac{1}{2} \times \frac{4}{7} \cdot \frac{4}{3} = \frac{8}{21}$$

PROBLEM 1.6

Find the equation of the straight line through $P(7, 5)$ perpendicular to the straight line *AB* whose equation is $3x + 4y - 16 = 0$. Calculate the length of the perpendicular from *P* and *AB*.

Solution:

Let Q be the point intersection of PQ and AB.
The slope of the line AB is $3x + 4y - 16 = 0$. ...(1)



$$4y = -3x + 16$$

$$y = \frac{-3}{4}x + 4$$

$$mAB = \frac{-3}{4}$$

The slope of the line $PQ = \frac{1}{mAB} = \frac{4}{3}$.

$$mPQ = \frac{y - y_1}{x - x_1}$$

$$\frac{4}{3} = \frac{y-5}{x-7} => 4x - 28 = 3y - 15$$

$$3y - 4x + 28 - 15 = 0$$

$$3y - 4x + 13 = 0 \qquad \text{...(2)}$$

Solving Equations (1) and (2), we obtain

$$3x + 4y - 16 = 0$$

$$3y - 4x + 13 = 0$$

From Equation (1), $\qquad 4y = -3x + 16$

$\Rightarrow \qquad y = -\frac{3}{4}x + 4$

By substituting this answer into Equation (2), we obtain

$$\left.\begin{aligned} 3\left(\frac{-3}{4}x+4\right)-4x+13&=0\\ x\left(\frac{9}{4}-4\right)+25&=0\\ x\left(\frac{\frac{-9}{4}+16}{4}\right)&=-25\\ \frac{-25}{4}x=-25 \;\therefore\; x&=4 \end{aligned}\right\}$$

By substituting this answer into Equation (1), we obtain $\frac{-9}{4}x+12-4x+13.$

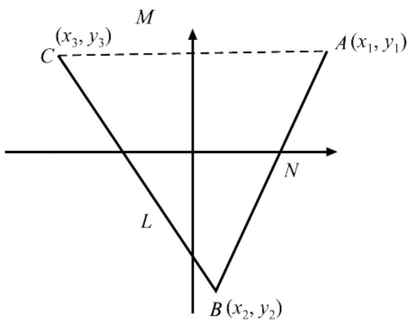
$$3(4)+4y-16=0$$
$$12+4y=16$$
$$4y=16-12$$
$$4y=4$$
$$y=1 \;\therefore\; Q(4,1)$$

$$PQ=\sqrt{(7-4)^2+(5-1)^2}=5 \qquad \textbf{(Ans. } 3y-4x+13=0;\,5)$$

PROBLEM 1.7

$L(-1,0)$, $M(3,7)$, and $N(5,-2)$ are the mid-points of the sides BC, CA, and AB, respectively, of the triangle ABC. Find the equation of AB.

Solution: The coordinates of A, B, and C are



$A(x_1, y_1)$

$B(x_2, y_2)$

$C(x_3, y_3)$

$$\frac{x_1+x_2}{2}=5 \qquad \ldots(1)$$

$$\frac{x_2+x_3}{2}=-1 \qquad \ldots(2)$$

$$\frac{x_1 + x_3}{2} = 3 \quad \ldots(3)$$

$$\frac{y_1 + y_2}{2} = -2 \quad \ldots(4)$$

$$\frac{y_2 + y_3}{2} = 0 => y_2 = -y_3 \quad \ldots(5)$$

$$\frac{y_1 + y_3}{2} = 7 \quad \ldots(6)$$

From (2), $x_2 + x_3 = -2$

From (3), $x_1 + x_3 = 6 \quad \ldots(7)$

Subtracting, we obtain $x_2 - x_1 = -8 \quad \ldots(8)$

$\therefore$ or $x_2 = x_1 - 8 \quad \ldots(9)$

By substituting this answer into Equation (1), we obtain

$$x_1 + x_2 = 10 => x_1 + (x_1 - 8) = 10$$

$$2x_1 = 18 \qquad \therefore \qquad x_1 = 9$$

By substituting this answer into Equation (7), we obtain

$$x_1 + x_3 = 6 => 9 + x_3 = 6 \therefore\ x_3 = -3$$

$$x_2 + x_3 = -2 => x_2 - 3 = -2$$

$\therefore$ $x_2 = 1$

Now as $y_1 + y_2 = -4;\ \ y_2 = -y_3$ and $y_1 + y_3 = 14$

Solving,
we obtain the following $y_1 = 5, y_2 = 9$ and $y_3 = 9$ (From (4), (5), and (6))

$\therefore$ $A(9, 5), B(1, -9)$ and $C(-3, 9)$

Slope of line $AB(m) = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-9-5}{1-9} = \dfrac{14}{-8} = \dfrac{14}{8}$

$$y - y_1 = m(x - x_1)$$

$$y - 5 = \frac{14}{8}(x - 9) => 8(y - 5) = 14(x - 9)$$

$$8y - 40 = 14x - 126$$

$$4y = 7x - 43$$

PROBLEM 1.8

The straight line $x - y - 6 = 0$ cuts the curve $y^2 = 8x$ at P and Q. Calculate the length of PQ.

Solution:

$$x - y - 6 = 0$$

$$y^2 = 8x$$

$$\left.\begin{aligned} x - y - 6 = 0 \ldots\ldots(1) \\ y^2 = 8x \ldots\ldots(2) \end{aligned}\right\} x - 6 = y => x = y + 6$$

$$\therefore \quad y^2 = 8x => y^2 = 8(y+6) => y^2 = 8y + 48$$

$$y^2 - 8y - 48 = 0 \Rightarrow \ y^2 - 12y + 4y - 48 = 0$$

$$\Rightarrow \quad (y-12)(y+4) = 0$$

$$\Rightarrow \quad y - 12 = 0,\ y = 12,\ (y-4) = 0,\ y = -4$$

When $\quad y = 12 \Rightarrow x - 12 - 6 = 0\ x = 18 \qquad \therefore\ p(18, 12)$

When $\quad y = -4 \Rightarrow x + 4 - 6 = 0 \Rightarrow x = 2 \qquad \therefore\ p(2, -4)$

$$\therefore \quad \text{Length } PQ = \sqrt{(18-2)^2 + (12-(-4))^2} = 16\sqrt{2}$$

PROBLEM 1.9

A line is drawn through the point (2, 3) making an angle of 45° with the positive direction of the *x*-axis, and it meets the line *x* = 6 at *P*. Find the distance of *P* from the origin *O* and the equation of the line through *P* perpendicular to *OP*.

Solution: Let us consider the point where the line through point (2, 3) and the line $x = 6$ meets at point $(6, y)$.

The slope of the line passing through (2, 3) and (6, y) is $\tan 45^\circ = \dfrac{3-y}{2-6} \Rightarrow 1 = \dfrac{3-y}{-4} \Rightarrow y = 7.$

The distance between the origin and the point $(6, y) = (6, 7)$ is

$$\sqrt{(6-0)^2 + (7-0)^2} = \sqrt{85}$$

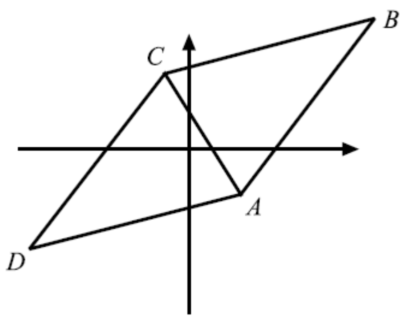
To find the equation of the through P and perpendicular to OP, we solve the following:

$$y - y_1 = m(x - x_1)$$
$$y - 7 = -\frac{6}{7}(x - 6) \qquad \left[\because \text{ Slope of } OP = \frac{7-0}{6-0} = \frac{7}{6}\right]$$
$$7y - 49 = -6x + 36$$
$$7y + 6x = 85$$

PROBLEM 1.10

The vertices of a quadrilateral *ABCD* are *A*(4, 0), *B*(14, 11), *C*(0, 6), and *D*(−10, − 5). Prove that the diagonals *AC* and *BD* bisect each other at right angles, and that the length of *BD* is four times that of *AC*.

Solution: $A(4, 0)$, $B(14, 11)$, C(9, 6), and $D(-10, -5)$ are the vertices of quadrilateral *ABCD*.



$$\therefore \qquad \text{The mid-point of } AC = \frac{4+0}{2} = 2; \qquad \frac{6+0}{2} = 3 \text{ is } (2, 3).$$
$$\text{The mid-point of } BD = \frac{14-10}{2} = 2; \qquad \frac{11-5}{2} = 3(2, 3).$$
$$\text{The slope of } AC(mAC) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-0}{0-4} = \frac{6}{-4} = -\frac{3}{2}.$$
$$\text{The slope of } BD(mBD) = \frac{-5-11}{-10-14} = \frac{-16}{-24} = \frac{2}{3}.$$
$$mAB \times mBC = \frac{-3}{2} \cdot \frac{2}{3} = -1 \Rightarrow AC \perp BD$$

$$AC = \sqrt{(4-0)^2 + (0-6)^2}$$
$$= 2\sqrt{13}$$
$$BD = \sqrt{(14+10)^2 + (11+5)^2}$$
$$= 8\sqrt{13}$$
$$\Rightarrow \qquad BD = 4AC$$

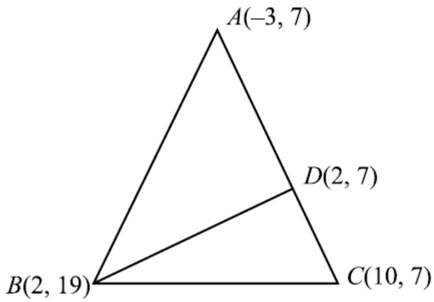
PROBLEM 1.11

The coordinates of vertices *A*, *B* and *C* of the triangle *ABC* are (−3, 7), (2, 19), and (10, 7), respectively.

(*a*) Prove that the triangle is isosceles.

(*b*) Calculate the length of the perpendicular from *B* to *AC*, and use it to find the area of the triangle.

Solution:



(*a*)
$$AB = \sqrt{(-3-2)^2 + (7-19)^2}$$
$$= \sqrt{25+144} = \sqrt{169} = 13$$
$$AC = \sqrt{(-3-10)^2 + (7-7)^2}$$
$$\sqrt{(-13)^2} = \sqrt{169} = 13$$

$\therefore$ $AB = AC \Rightarrow$ the triangle is isosceles

(*b*) Let D be the bisection between B and AC, D (2, 7).

$$BD = \sqrt{(2-2)^2 + (7-19)^2} = 12$$

Area of the triangle $= \frac{1}{2}(BD \times AC) = \frac{1}{2} \times 12 \times 13 = 78$

PROBLEM 1.12

Find the equations of the lines that pass through the point of intersection of the lines $x-3y=4$ and $3x+y=2$, and are respectively parallel and perpendicular to the line $3x+4y=0$.

Solution: $\left.\begin{array}{l} x-3y=4\ldots\ldots(1) \\ 3x+y=2\ldots\ldots(2) \end{array}\right\} x=4+3y$

By substituting this answer into Equation (2), we obtain

$$3(4+3y)+y=2 \quad\Rightarrow\quad 12+9y+y=2$$
$$10y=2-12 \quad\Rightarrow\quad 10y=-10$$
$$\therefore \quad y=-1$$

By substituting y into Equation (1), we obtain

$$x+3=4\Rightarrow x=4-3=1 \quad \therefore \quad x=1$$

$\therefore$ The point of intersection between two lines is $(1,-1)$.

The slope of line $3x+4y=0$ is $m=\frac{-3}{4}$.

The equation of a parallel line is $=\frac{-3}{4}$ and point $(1,-1)$.

$\therefore$ The equation of the required line is as follows:

$$y-y_1=m(x-x_1)$$
$$y+1=\frac{-3}{4}(x-1)\Rightarrow 4y+3x+1=0$$

The equation of perpendicular line is $m=\frac{4}{3}$ and point $(1,-1)$.

$$\Rightarrow \quad y+1=\frac{4}{3}(x-1)$$
$$\Rightarrow \quad 3y+3=4x-4$$
$$\Rightarrow \quad 3y-4x+7=0$$

PROBLEM 1.13

Through the point A (1, 5), a line is drawn parallel to the *x*-axis to meet the line *PQ* at *B*, whose equation is $3y=2x-5$. Find the length of *AB* and the sine of the angle between *PQ* and *AB*; hence, show that the length of the perpendicular from *A* to *PQ* is $18/\sqrt{13}$. Calculate the area of the triangle formed by *PQ* and the axes.

Solution: The line *PQ* is

$$3y = 2x - 5$$

$$3y = 2x - 5 \qquad \ldots(1)$$

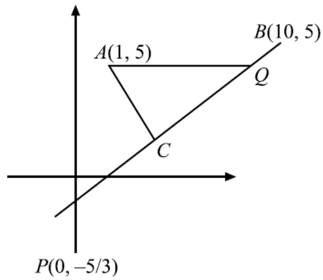
$$y = 5 \qquad \ldots(2)$$

(Since line AB ∥ to x-axis)

$$3 \times 5 = 2x - 5$$

$$15 = 2x - 5$$

$$20 = 2x \qquad \therefore \quad x = 10$$



$\therefore\ B(10, 5)$ $$AB = \sqrt{(10-1)^2 + (5-5)^2} = 9$$

$$\theta = 45 \Rightarrow m = \tan\theta$$

From Equation (1) $$3y = 2x - 5$$

$$y = \frac{2}{3}x - \frac{5}{3}$$

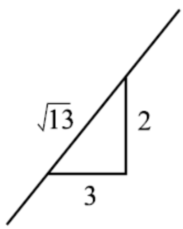
$$y - y_1 = m(x - x_1)$$

$$m = \frac{2}{3}$$

$$m = \tan\theta$$

$$\frac{2}{3} = \tan\theta \Rightarrow \qquad \theta = \tan^{-1}\frac{2}{3} = 33.4$$

$\because$ $$m = \frac{2}{3} \quad \Rightarrow \qquad \sin\theta = \frac{2}{\sqrt{13}}$$



$$\sqrt{2^2 + 3^2} = \sqrt{13}$$

$$\sin\theta = \frac{2}{\sqrt{13}}$$

$$3y = 2x - 5 \Rightarrow PQ$$

$$2y = 3x + 13 \Rightarrow AC$$

The slope of $PQ(mPQ) = \dfrac{2}{3} \Rightarrow AC \perp PQ$.

$\therefore$ The slope of $AC(mAC) = \frac{-3}{2}$ and $A(1, 5)$ is as follows:

$$y - y_1 = m(x - x_1)$$

$$y - 5 = \frac{-3}{4}(x - 1) \Rightarrow 2y + 3x - 13 = 0$$

$$2y + 3x - 13 = 0 \qquad \ldots(1)$$

$$3y = 2x - 5 \qquad \ldots(2)$$

From Equation (1),

$$2y + 3x - 13 = 0 \Rightarrow 2y = 13 - 3x \Rightarrow y = \frac{13}{2} - \frac{3}{2}x$$

By substituting y into Equation (2), we obtain

$$2y + 3x - 13 = 0 \Rightarrow 2y = 13 - 3x \Rightarrow y = \frac{13}{2} - \frac{3}{2}x$$

By substituting y into Equation (2), we obtain

$$3\left(\frac{13}{2} - \frac{3}{2}x\right) = 2x - 5 \therefore x = \frac{49}{13}$$

By substituting this answer into Equation (2), we obtain

$$\sqrt{\frac{4212}{(13)}} \div 13 = -5 \;\; 3y = 2\left(\frac{49}{13}\right) - 5$$

$$-x\left(\frac{13}{2}\right) = \frac{-39}{2} - 5 \;\; 3y = \frac{98}{13} - 5$$

$$-x\left(\frac{13}{2}\right) = \frac{-39 - 10}{2} \;\; 3y = \frac{98 - 63}{13}$$

$$-x\left(\frac{13}{2}\right) = \frac{-49}{2} \;\; 3y = \frac{33}{13} \;\therefore\; y = \frac{11}{13}$$

$$\therefore\; C\left(\frac{49}{13}, \frac{11}{13}\right) \qquad AC = \sqrt{\left(1 - \frac{49}{13}\right)^2 + \left(5 - \frac{11}{13}\right)^2} = \frac{18}{\sqrt{13}}$$

$$= \sqrt{\frac{(-36)^2}{(13)^2} + \frac{(54)^2}{(13)^2}} \rightarrow \sqrt{\frac{1296}{(13)^2} + \frac{2916}{(13)^2}} = \sqrt{\frac{4212}{(13)}} \div 13$$

$$\Rightarrow \qquad \sqrt{\frac{323}{13}} \Rightarrow \frac{\sqrt{324}}{\sqrt{13}} = \frac{18}{\sqrt{13}}$$

To find the intersection PQ to the axis,

At $\quad y = 0, \quad 3y = 2x - 5$

$\Rightarrow \quad x = \frac{5}{2}\left(\frac{5}{2}, 0\right)$

At $\quad y = 0 \Rightarrow x = \frac{-5}{3}\left(0, \frac{-5}{3}\right)$

$\therefore \quad \text{Area} = \frac{1}{2} \times \frac{5}{3} \times \frac{5}{2} = \frac{25}{12}$

PROBLEM 1.14

Let $y = \frac{x^2 + 2}{x^2 - 1}$ express x in terms of y and find the values of y for which x is real.

Solution:

$$y = \frac{x^2 + 2}{x^2 - 1}$$

$$y(x^2 - 1) = x^2 + 2$$

$$yx^2 - y = x^2 + 2$$

$$yx^2 - x^2 = y + 2$$

$$x^2(y - 1) = y + 2$$

$$\therefore \quad x^2 = \frac{y + 2}{y - 1}$$

For x real numbers, $\quad \frac{y + 2}{y - 1} \geq 0$

$$D_x = \forall_x : \geq 0$$

$R_y : \leq -2 \text{ or } y > 1$

$$\therefore \quad x = \pm\sqrt{\frac{y + 2}{y - 1}}$$

PROBLEM 1.15

Find the domain and range of each function:

$(a)\ y = \frac{1}{1 + x^2}, \qquad (b)\ y = \frac{1}{1 + \sqrt{x}}, \qquad (c)\ y = \frac{1}{\sqrt{3 - x}}$

Solution: (*a*) $y = \dfrac{1}{1+x^2}$

$$y(1+x^2) = 1 \Rightarrow y + yx^2 = 1$$

$$yx^2 = 1 - y \therefore x^2 = \frac{1-y}{y}$$

$$\therefore \quad x = \pm\sqrt{\frac{1-y}{y}}$$

$$D_x \forall\, x$$

$$R_y : 0 < y \le 1$$

(*b*) $y = \dfrac{1}{1+\sqrt{x}}$

$$y\left(1+\sqrt{x}\right) = 1$$

$$y + y\sqrt{x} = 1$$

$$y\sqrt{x} = 1 - y \rightarrow$$

$$\sqrt{x} = \frac{1-y}{y}$$

$$x = \left(\frac{1-y}{y}\right)^2 \Rightarrow x = \left(\frac{1}{y} - 1\right)^2 \; D_x \forall\, x \ge 0$$

$$\therefore \quad R_y : y > 0$$

(*c*) $y = \dfrac{1}{\sqrt{3-x}}$

$$y\left(\sqrt{3-x}\right) = 1$$

$$\sqrt{3-x} = \frac{1}{y}$$

$$3 - x = \left(\frac{1}{y}\right)^2 \text{ for real numbers}$$

$$\therefore \quad x = 3 - \left(\frac{1}{y}\right)^2 \; D_x \forall\, x < 3$$

$$R_y : \forall\, y > 0$$

PROBLEM 1.16

Find the points of intersection of $x^2 = 4y$ and $y = 4x$.

Solution:

$$x^2 = 4y \quad \ldots(1)$$
$$y = 4x \quad \ldots(2)$$

$\Rightarrow$ $x^2 = 4(4x) \Rightarrow x^2 = 16x$

$x^2 - 16x = 0$

$x = 0$ or $x = 16$ when $x = 0,\ y = 0$

and when $x = 16,\ y = 64$

$\therefore$ The points of intersection are $(0, 0)(16, 64)$.

PROBLEM 1.17

Find the coordinates of the points at which the curves cut the axes:
(*a*) $y = x^3 - 9x^2$, (*b*) $y = (x^2 - 1)(x^2 - 9)$, (*c*) $y = (x+1)(x-5)^2$

Solution: (*a*) $y = x^3 - 9x^2$

At $x = 0, y = 0,$ the curve cuts y-axix at $(0, 0)$

At $y = 0 \Rightarrow 0 = x^3 - 9x^2$

$x^2(x-9) = 0$ either $x^2 = 0$

or $(x-9) = 0$ $\therefore$ $x = 9$

$\therefore$ The curves cut the x-axis at $(0, 0)(9, 0)$.

(*b*) $y = (x^2 - 1)(x^2 - 9)$

At $x = 0$ $y = (0-1)(0-9) \Rightarrow y = (-1)(-9) = 9$

$\therefore$ The curves cut the y-axis at $(0, 9)$.

At $y = 0 \Rightarrow 0 = (x^2 - 1)(x^2 - 9)$

either $x^2 - 1 = 0$ $x \Rightarrow 2 = 1$ $\therefore$ $x = \pm 1$

or $x^2 - 9 = 0 \Rightarrow x^2 = 1$ $\therefore$ $x = \pm 3$

$\therefore$ The curve cuts the x-axis at $(1, 0), (-1, 0)$ or $(3, 0), -(-3, 0)$.

(*c*) $y = (x+1)(x-5)^2$

At $x = 0, y = (0+1)(0-5)^2 \Rightarrow y = (1)(25) = 25$

$\therefore$ The curves cut the y-axis at $(0, 25)$.

At $\quad y = 0 \Rightarrow (x+1)(x-5)^2$

either $\quad x+1=0 \Rightarrow x=-1$

or $\quad (x-5)^2 = 0 \Rightarrow x = 5$

$$(x-5)^2 = 0$$

$$x-5=0 \Rightarrow x=5$$

$\therefore$ The curves cut the x-axis at (–1, 0) or (5, 0).

PROBLEM 1.18

Let $f(x) = ax + b$ and $g(x) = cx + d$. What condition must be satisfied by the constants a, b, c, and d to make $f[g(x)]$ and $g[f(x)]$ indentical?

Solution:

$$f(g(x)) = f(cx+d) = a(cx+d)+b$$

$$g(f(x)) = g(ax+b) = c(ax+b)+d$$

since $\quad f(g(x)) = g(f(x)) = a(cx+d)+b = c(ax+b)+d$

$\Rightarrow \quad acx + ad = +b\ cax + cd + d$

$ad + b = cb + d$ is the required condition.

PROBLEM 1.19

A particle moves in the plane from (–2, 5) to the y-axis in such a way that $\Delta y = 3 * \Delta x$. Find its new coordinates.

Solution: $(-2, 5)\ \Delta y = 3\Delta x$

$$\Delta x = x_2 - x_1 \Rightarrow 0-(-2) = 2$$

$\therefore \quad \Delta y = 3 \times 2 = 6$

$$\Delta y = y_2 - y_1$$

$$6 = y_2 - y_1$$

$\therefore \quad y_2 = 6 + y_1$

$\therefore \quad y_2 = 6 + y_1$

$\therefore \quad y_2 = 6 + 5 = 11$

Hence (0, 11) is the new coordinate in the (+) y-axis.

$$\text{Again} \quad \Delta x = -2 - 0 = -2 \Rightarrow \Delta y = 3\,\Delta x = 3 \times -2 = -6$$

$$\therefore \quad \Delta y = y_2 - y_1$$

$$\therefore \quad y_2 = \Delta y + y_1 \Rightarrow y_2 = -6 + 5 = -1$$

Hence $(0,-1)$ is the new coordinate in the $(-)$ y-axis.

PROBLEM 1.20

If $f(x) = 1/x$ and $g(x) = 1/\sqrt{x}$, what are the domains of f, g, $f+g$, $f-g$, $f{\cdot}g$, f/g, g/f, fog, and gof? What is the domain of $h(x) = g(x+4)$?

Solution: $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{1}{\sqrt{x}}$

$$(a) \quad f(x) \Rightarrow \frac{1}{x} D_x \ \forall\, x \neq 0$$

$$(b) \quad g(x) \Rightarrow \frac{1}{\sqrt{x}} D_x \ \forall\, x > 0$$

$$(c) \quad f + g \Rightarrow \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}} D_x \forall x > 0$$

$$(d) \quad f - g \Rightarrow \frac{1}{x} - \frac{1}{\sqrt{x}} D_x \forall x > 0$$

$$(e) \quad f \times g \Rightarrow \frac{1}{x} \times \frac{1}{\sqrt{x}} D_x \forall x > 0$$

$$(f) \quad \frac{f}{g} \Rightarrow \frac{(1/x)}{(1\sqrt{x})} = \frac{1}{\sqrt{x}} Dx\forall\_x > 0$$

$$(g) \quad \frac{g}{f} \Rightarrow \frac{\left(\frac{1}{\sqrt{x}}\right)}{\left(\frac{1}{x}\right)} = \frac{1}{\sqrt{x}} \Rightarrow \frac{1}{\sqrt{x}} \cdot \sqrt{x} \cdot \sqrt{x} \Rightarrow \sqrt{x} D_x \forall\_x \geq 0$$

$$(h) \quad fog \Rightarrow f(g(x)) = f\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\frac{\sqrt{1}}{\sqrt{x}}} = \sqrt{x} D_x \forall x \geq 0$$

$$(i) \quad gof \Rightarrow g(f(x)) = g\left(\frac{1}{x}\right) = \frac{\frac{1}{1}}{\sqrt{x}} = \sqrt{x} D_x \forall x \geq 0$$

$$(j) \quad h(x) = g(x+4) = g\frac{1}{\sqrt{x+4}} D_x \forall x > -4$$

PROBLEM 1.21

Discuss the continuity of the function.

$$f(x)=\begin{cases} x+\dfrac{1}{x} & \text{for} \quad x<0 \\ -x^3 & \text{for} \quad 0\le x<1 \\ -1 & \text{for} \quad 1\le x<2 \\ 1 & \text{for} \quad x=2 \\ 0 & \text{for} \quad x>2 \end{cases}$$

Solution: At $x=0 \to f(-x^3)=0.$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} f\left(x+\frac{1}{x}\right)=\infty$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} f(-x^3)=0 \neq \lim_{x\to 0^-}$$

$\therefore$ The function is not continuous at $x=0$.

At $\qquad x=1 \Rightarrow f(+1)=-1$

$$\lim_{x\to -1} f(x) \Rightarrow \lim_{x\to +1} -(-x^3)=-1$$

$$\lim_{x\to +1} f(x) \Rightarrow \lim_{x\to x+1} (-1)=-1$$

$\therefore \quad \lim_{x\to -1} f(x) \Rightarrow \lim_{x\to +1} f(x)$

$\therefore$ The function is continuous at $x=1$.

At $\qquad x=2, f(2)=1$

$$\lim_{x\to -2} f(x) \Rightarrow \lim_{x\to -2} (-1)=-1$$

$\therefore$ The function is not continuous at $x=2$.

PROBLEM 1.22

Evaluate the following limits:

(*a*) $\lim\limits_{x\to\infty} \dfrac{x+\sin x}{2x+5}$

(*b*) $\lim\limits_{x\to\infty} \dfrac{1+\sin x}{x}$

(*c*) $\lim\limits_{x\to 0} \dfrac{x}{\tan 3x}$

(*d*) $\lim\limits_{x\to\infty 0} \dfrac{x\sin x}{(x+\sin x)^2}$

(e) $\lim\limits_{x\to 1}\dfrac{1-\sqrt{x}}{1-x}$

(f) $\lim\limits_{x\to 1}\dfrac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}$

(g) $\lim\limits_{n\to\infty}\left(\sqrt{n^2+1-n}\right)$

Solution: (*a*) $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{2x+5} \div \lim\limits_{x\to\infty}\left(\dfrac{1+\dfrac{\sin x}{x}}{2+\dfrac{5}{x}}\right) \Rightarrow \dfrac{1+0}{2+\dfrac{5}{\infty}}$

or $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{2x+5} + \lim\limits_{x\to\infty}\left(\dfrac{1}{x}+\dfrac{\sin x}{x}\right) = \dfrac{1}{2}$

(*b*) $\lim\limits_{x\to\infty}\dfrac{1+\sin x}{x} \Rightarrow \lim\limits_{x\to\infty}\left(\dfrac{1}{x}+\dfrac{\sin x}{x}\right) = \dfrac{1}{\infty}+\dfrac{\sin\infty}{\infty} = 0$

(*c*) $\lim\limits_{x\to 0}\dfrac{x}{\tan 3x} \Rightarrow \lim\limits_{x\to 0} \Rightarrow \lim\limits_{x\to 0}\dfrac{x}{\sin 3x}\cdot\cos 3x$

$$\Rightarrow \quad \lim_{x\to 0}\cos 3x\cdot\frac{1}{\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}} \Rightarrow \lim_{x\to 0}\cos 3x\cdot\frac{1}{\dfrac{3}{3}\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}}$$

$$\lim_{x\to 0}\cos 3x\cdot\frac{1}{3\lim\limits_{x\to 0}\cdot\dfrac{\sin 3x}{x}} \Rightarrow \cos(0)\cdot\frac{1}{3\times 1} = 1\cdot\frac{1}{3} = \frac{1}{3}$$

(*d*) $\lim\limits_{x\to\infty}\dfrac{x\sin x}{(x+\sin x)^2}$

$$\lim_{x\to\infty}\frac{x}{x+\sin x}\lim_{x\to\infty}\frac{\sin x}{x+\dfrac{\sin x}{x}} \div x$$

$$\lim_{x\to\infty}\frac{1}{1+\dfrac{\sin x}{x}}\cdot\lim_{x\to\infty}\frac{x}{1+\dfrac{\sin x}{x}}$$

$$= \frac{1}{1+\dfrac{\sin\infty}{\infty}}\cdot\frac{\dfrac{\sin}{\infty}}{1+\dfrac{\sin\infty}{\infty}}$$

$$= \frac{1}{1+0}\cdot\frac{0}{1+0} = 0$$

$$(e)\quad \lim_{x\to 1}\frac{1-\sqrt{x}}{1-x} \Rightarrow \lim_{x\to 1}\frac{1-\sqrt{x}}{\left(1-\sqrt{x}\right)\left(1+\sqrt{x}\right)}$$

$$\Rightarrow \qquad \lim_{x\to 1}\frac{1}{\left(1+\sqrt{x}\right)} = \frac{1}{\left(1+\sqrt{x}\right)} = \frac{1}{1+1} = \frac{1}{2}$$

$$(f)\quad \lim_{x\to 1}\frac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}\times\frac{\sqrt{x+1}+\sqrt{2x}}{\sqrt{x+1}+\sqrt{2x}}$$

$$\lim_{x\to 1}\frac{x+1-2x}{(x^2-x)\left(\sqrt{x+1}-\sqrt{2x}\right)}$$

$$\lim_{x\to 1}\frac{x+1-2x}{x(x-1)\left(\sqrt{x+1}+\sqrt{2x}\right)} = \lim_{x\to 1}\frac{(1-x)}{x(x-1)\left(\sqrt{x+1}+\sqrt{2x}\right)}$$

$$\Rightarrow \qquad = -\lim_{x\to 1}\frac{1}{x\left(\sqrt{x+1}+\sqrt{2x}\right)} = \frac{-1}{1\sqrt{2}+\sqrt{2}} = -\frac{1}{2\sqrt{2}}$$

$$(g)\quad \lim_{n\to\infty}\left(\sqrt{n^2+1}-n\right)\times\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$$

$$\Rightarrow \qquad \lim_{n\to\infty}\left[\sqrt{n^2+1}-n\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}\right]$$

$$\lim_{n\to\infty}\frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$$

$$\Rightarrow \qquad \lim_{n\to\infty}\frac{1}{\sqrt{n^2+1}-n} = \frac{1}{\sqrt{\infty^2+1}+\infty} = \frac{1}{\infty} = 0$$

PROBLEM 1.23

Suppose that $f(x) = x^3 - 3x^2 - 4x + 12$ **and** $h(x) = \begin{Bmatrix} \dfrac{f(x)}{x-3} & \text{for } x \neq 3 \\ k & \text{for } x = 3 \end{Bmatrix}$. **Find**

(*a*) all zeros of *f*

(*b*) the value of *k* that makes *h* continuous at $x = 3$.

Solution:

$(a)\ x^3 - 3x^2 - 4x + 12 = 0 \Rightarrow x^2(x-3) - 4(x-3) = 0 \Rightarrow (x^2-4)(x-3) = 0$

either $x^2 - 4 = 0 \Rightarrow x = \pm 2$

or $x - 3 = 0 \Rightarrow x = 3$

$(b)\ \dfrac{f(x)}{x-3} = \dfrac{x^3 - 3x^2 - 4x + 12}{x-3} = \dfrac{(x^2-4)(x-3)}{(x-3)}$

$= x^2 - 4$

$h(3) = (3)^2 - 4 \Rightarrow 9 - 4 = 5$

$k = 5$

CHAPTER 2

FUNCTIONS

PROBLEMS

PROBLEM 2.1

A body of unknown temperature was placed in a room that was held at 30°F. After 10 minutes, the body's temperature was 0°F, and 20 minutes after the body was placed in the room, the body's temperature was 15°F. Use Newton's law of cooling to estimate the body's initial temperature.

Solution:

$T - T_s = (T_o - T_s)e^{tk}$ when $t = 10$ minute $T = 0$°F we have

$$0 - 30 = (T_o - 30)e^{10k}$$

$$\Rightarrow \qquad -30 = (T_0 - 30)e^{10k} \qquad \ldots(1)$$

When $t = 20, T = 15$°F, we have

$$15 - 30 = (T_o - 30)e^{20k} \qquad \ldots(2)$$

$$-15 = (T_0 - 30)e^{20k}$$

Divide Equation (2) by Equation (1):

$$e^{10k} = \frac{1}{2} \qquad \ldots(3)$$

By substituting Equation (3) into Equation (1), we obtain

$$-30 = (T_o - 30)\left(\frac{1}{2}\right)$$

$$\therefore \qquad T_o - 30 = -\frac{30}{1/2} = -60$$

$$\therefore \qquad T_o - 60 + 30 = -30\,F$$

PROBLEM 2.2

A pan of warm water (46°C) was put in a refrigerator. Ten minutes later, the water's temperature was 39°C, and 10 minutes after that, it was 33°C. Use Newton's law of cooling to estimate how cold the refrigerator was. **Ans.** $3°C$

Solution:

$$T - T_s = \left(T_o - T_s\right)e^{tk}$$

When, $T_o = 46°\text{C}, t = 10\,\text{min}, T = 39°\text{C}$, we have

$$39 - T_s = \left(46 - T_s\right)e^{10k} \qquad \ldots(1)$$

When $t = 20\,\text{min}, T = 33°\text{C}$, we have

$$33 - T_s = \left(46 - T_s\right)e^{20k} \qquad \ldots(2)$$

By dividing Equation (2) by Equation (1), we obtain

$$\frac{33 - T_s}{39 - T_s} = e^{10k}$$

By substituting this answer into Equation (1), we obtain

$$39 - T_s = \left(46 - T_s\right)\left(\frac{33 - T_s}{39 - T_s}\right)$$

$$\left(39 - T_s\right)\left(39 - T_s\right) = \left(46 - T_s\right)\left(33 - T_s\right)$$

$$1521 - 39T_s - 39 + T_s^2 = 1518 - 46T_s - 33T_s + T_s^2$$

$$\Rightarrow \qquad 1521 - 78T_s + T_s^2 - 1518 - 79T_s - T^2s = 0$$

$$3 + T_s = 0 \therefore T_s = -3$$

PROBLEM 2.3

Solve the following equations for values of θ from –180° to 180° inclusive:

(i) $\mathbf{\tan^2\theta + \tan\theta = 0}$

(ii) $\mathbf{\cot\theta = 5\cos\theta}$

(iii) $\mathbf{3\cos\theta + 2\sec\theta + 7 = 0}$

(iv) $\mathbf{\cos^2\theta + \sin\theta + 1 = 0}$

Solution: (*i*) $\tan^2\theta + \tan\theta = 0 \quad \Rightarrow \quad \tan\theta(\tan\theta + 1) = 0$

either $\tan\theta = 0$ $\qquad (\therefore \theta = -180°, 0°, 180°)$

or $\tan\theta + 1 = 0$

$\therefore \quad \tan\theta = -1$ $\qquad (\therefore \theta = -45°, 135°)$

(*ii*) $\cot\theta = 5\cos\theta \Rightarrow \dfrac{\cos\theta}{\sin\theta} = 5\cos\theta$

$$5\sin\theta\cos\theta = \cos\theta$$

$$5\cos\theta\sin\theta - \sin\theta - 1$$

$$\cos\theta(5\sin\theta - 1)$$

either $\cos\theta = 0 \Rightarrow \theta = 90°, -90°$

or $\quad 5\sin\theta - 1 = 0 \Rightarrow \sin\theta = 1 \therefore \sin\theta = \dfrac{1}{5}$

$\therefore \quad \theta = 11.54168.46$

$\therefore \quad \theta = \{90°, 90°, 11.54°, 168.46°\}$

(*iii*) $3\cos\theta + 2\dfrac{1}{\cos\theta} + 7 = 0 \Rightarrow 3\cos^2\theta + 7\cos\theta + 2 = 0$

$\Rightarrow \quad (3\cos\theta + 1)(\cos\theta + 2) = 0$

or $\quad$ either $\cos\theta = -2 \Rightarrow$ neglected as $-1 \le \cos\theta \le 1$

or $\quad 3\cos\theta + 1 = 0 \quad \Rightarrow \quad \cos\theta = -\dfrac{1}{3}$

$\Rightarrow \quad \theta = \{-109.47°, 109.47°\}$

(*iv*) $1 - \sin^2\theta + \sin\theta + 1 = 0 \Rightarrow (\sin\theta - 2)(\sin\theta + 1) = 0$

either $\sin\theta = -2$ neglected as $-1 \le \sin\theta \le 1$

or $\quad \sin\theta = -1 \Rightarrow \theta = -90°$

$\therefore \quad \theta = \{-90°\}$

PROBLEM 2.4

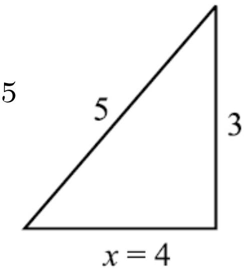
Solve the following equations for values of θ from 0° to 360° inclusive:

(*i*) $3\cos 2\theta - \sin\theta + 2 = 0$

(*ii*) $3\tan\theta = \tan 2\theta$

(*iii*) $\mathbf{\sin 2\theta \cdot \cos\theta + \sin^2\theta = 1}$

(*iv*) $\mathbf{3 \cot 2\theta + \cot\theta = 1}$



Solution: (*i*) $3(1-2\sin^2\theta) - \sin\theta + 2 = 0 \quad \Rightarrow \quad 6\sin^2\theta + \sin\theta - 5$

$\Rightarrow \qquad (6\sin\theta - 5)(\sin\theta + 1) = 0$

$\Rightarrow \qquad \text{either } \sin\theta = \dfrac{5}{6}$

$\Rightarrow \qquad \theta = 56.4°, 123.6°$

or $\qquad \sin\theta = -1 \Rightarrow \theta = 270$

$\theta = \{56.4°, 123.6°, 270°\}$

(*ii*) $3\tan\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} \Rightarrow \tan\theta\left(3\tan^2 - 1\right) = 0$

either $\tan\theta = 0 \Rightarrow \theta = 0°, 180°, 360°$

Or $\qquad \tan\theta = (1/\sqrt{3}) \Rightarrow = 150°, 210°$

Or $\qquad \tan\theta = (1/\sqrt{3}) \Rightarrow \theta = 150°, 330°$

$\theta = \{0°, 30°, 150°, 180°, 210°, 330°, 360°\}$

(*iii*) $(2\sin\theta\cos\theta)\cos\theta + (1-\cos^2\theta) - 1 = 0$

$\Rightarrow \cos^2\theta\,(2\sin\theta - 1) = 0$

either $\cos^2\theta = 0 \Rightarrow \theta = 90°, 270°$

or $\qquad \sin\theta = 1/2 \quad \Rightarrow \quad \theta = 30°, 150°$

$\theta = \{30°, 90°, 150°, 270°\}$

(*iv*) $3\cot 2\theta + \cot\theta = 1$

$$\Rightarrow \qquad \frac{3}{\tan 2\theta} + \frac{1}{\tan\theta} = 1$$

$$\Rightarrow \qquad \frac{3}{\dfrac{2\tan\theta}{1-\tan^2\theta}} + \frac{1}{\tan\theta} = 1$$

$$\Rightarrow \frac{3(1-\tan^2\theta)}{2\tan\theta} + \frac{1}{\tan\theta} = 1$$

$$\Rightarrow \frac{3(1-\tan^2\theta)}{2\tan\theta} + \frac{1}{\tan\theta} = 1$$

$$\Rightarrow \frac{3-3\tan^2\theta+2}{2\tan\theta}=1 \Rightarrow 5-3\tan^2\theta=2\tan\theta$$

$$\Rightarrow 3\tan^2\theta+2\tan\theta-5=0$$

$$\Rightarrow 3\tan^2\theta+2\tan\theta-3\tan\theta-5=0$$

$$\Rightarrow \tan\theta\,(3\tan\theta+5)-1\,(3\tan\theta+5)=0$$

$$\Rightarrow (\tan\theta-1)-(3\tan\theta+5)=0$$

either $\tan\theta-1=0 \Rightarrow \tan\theta-1 \Rightarrow \theta=45°, 225°$

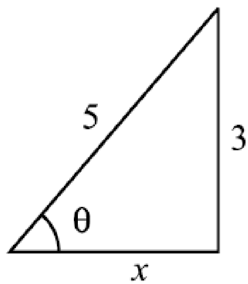
or $3\tan\theta+5=0 \Rightarrow \tan\theta=-\frac{5}{3} \Rightarrow \theta=121°, 301°$

$\therefore \quad \theta=[45°,\ 121°,\ 225°,\ 301°]$

PROBLEM 2.5

If sin θ = 3/5, find without using tables the value of
(*i*) cos θ
(*ii*) tan θ

Solution: $\sin\theta=\frac{3}{5}$



$\therefore \quad 5^2=x^2+3^2$

$25=x^2+9$

$x^2=25-9=16$

$\Rightarrow \quad x=4$

$\therefore \quad \cos\theta=\frac{4}{5}$ and $\tan\theta=\frac{3}{4}$

PROBLEM 2.6

Find, without using tables, the values of cos x and sin x, when cos $2x$ is $\frac{1}{8}$.

Solution: As $2\cos^2 x-1=\cos 2x=\frac{1}{8}$

$\therefore \quad 2\cos^2 x=\frac{1}{8}+1 \Rightarrow 2\cos^2 x=\frac{9}{8}$

$\therefore \quad \cos^2 x=\frac{9}{16}$

$$\cos x = \pm\frac{3}{4}$$

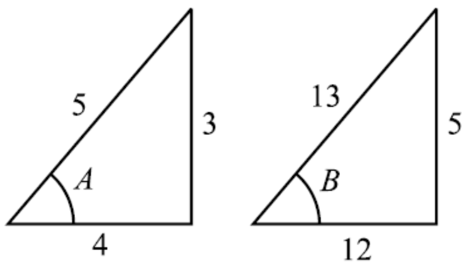
$$\sin^2 x = 1 - \cos^2 x \Rightarrow 1 - \left(\pm\frac{3}{4}\right)^2 = 1 - \frac{9}{16} = \frac{7}{16}$$

$$\therefore \qquad \sin x = \frac{\sqrt{7}}{4}$$

PROBLEM 2.7

If $\sin A = 3/5$ and $\sin B = \frac{5}{13}$, where A and B are acute angles, find, without using tables, the values of

(*a*) $\sin(A+B)$
(*b*) $\cos(A+B)$
(*c*) $\cot(A+B)$



Solution: $\sin A = \frac{3}{5}, \sin B = \frac{5}{13}$

(*a*) As we know that

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$= \frac{3}{5} \times \frac{12}{13} + \frac{4}{3} \times \frac{5}{13} = \frac{56}{63}$$

(*b*) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= \frac{4}{5} \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13} = \frac{33}{65}$$

(*c*) $\cot(A+B) \Rightarrow \frac{\cos(A+B)}{\sin(A+B)} = \frac{\frac{33}{65}}{\frac{56}{65}} = \frac{33}{56}$

PROBLEM 2.8

If $\tan A = -1/7$ and $\tan B = 3/4$, where A is obtuse and B is acute, find, without using tables, the value of $A - B$.

Solution:

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$$

$$= \frac{-\frac{1}{7} - \frac{3}{4}}{1 + \left(-\frac{1}{7}\right)\left(\frac{3}{4}\right)} \Rightarrow \tan(A-B) = 1$$

$$\therefore \qquad A - B = 135°$$

PROBLEM 2.9

Prove the following identities:

(*i*) $\sec^2\theta + \cos^2\theta = \sec^2\theta \cos^2\theta$

(*ii*) $\sin^2\theta\left(1 + \sec^2\theta\right) = \sec^2\theta - \cos^2\theta$

(*iii*) $\dfrac{1+\sin\theta}{1+\sin\theta} = (\sec\theta + \tan\theta)^2$

(*iv*) $\sec\theta - \sin\theta = \dfrac{\tan^2\theta + \cos^2\theta}{\sec\theta + \sin\theta}$

(*v*) $\dfrac{\cos(A-B) - \cos(A+B)}{\sin(A+B) + \sin(A-B)} = \tan B$

(*vi*) $\cos B - \cos A \cdot \cos(A-B) = \sin A \cdot \sin(A-B)$

(*vii*) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \times \tan B \times \tan C}{1 - \tan B \cdot \tan C - \tan C \cdot \tan A - \tan \cdot \tan B}$

If A, B, C are angles of a triangle, show that
$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$

(*viii*) $\dfrac{1}{2}[\tan(x+h) + \tan(x-h)] - \tan x = \dfrac{\tan x \cdot \sin^2 h}{\cos^2 x - \sin^2 h}$

(*ix*) $\tan x = \sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$

(x) $\dfrac{\sin 4A + \sin 2A}{\cos 4 + \cos 2A + 1} = \tan 2A$

(xi) $\sin^4\theta + \cos^4\theta = \dfrac{1}{4}(\cos 4\theta + 3)$

(xii) $4\sin^3 A \cdot \cos 3A + 4\cos^2 A \cdot \sin 3A = 3\sin 4A$

(xiii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

(xiv) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(xv) $\cot^{-1} x = \dfrac{\pi}{2} - \tan^{-1} x$

(xvi) $\cosh(u+v) = \cosh u \cdot \cosh u \cdot \cosh v + \sinh u \cdot \sinh v$ and then verify

$\cosh(u-v) = \cosh u \cdot \cosh(u+v) - \sinh(u-v)]$

(xvii) $\cosh u \cdot \sinh v = \dfrac{1}{2}[\sinh(u+v) - \sinh(u-v)]$

(xviii) $\sinh u \cdot \sinh v = \dfrac{1}{2}[\cosh(u+v) - \cosh(u-v)]$

(xix) $\cosh 3u = \cosh u + 4\sinh^2 u \cdot \cosh u = 4\cosh^3 u - \cosh u$

(xx) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

Solution: *(i)* $\sec^2\theta + \csc^2\theta = \sec^2\theta \csc^2\theta$

left-hand side $\Rightarrow \sec^2\theta + \cos^2\theta = \dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta \sin^2\theta}$

As $\sin^2\theta + \cos^2\theta = 1$

$$\therefore \quad \frac{1}{\cos^2\theta\sin^2\theta} = \frac{1}{\cos^2\theta}\cdot\frac{1}{\sin^2\theta} = \sec^2\theta\cos^2\theta = \text{right-hand side}$$

(ii) $\sin^2\theta(1 + \sec^2\theta) = \sec^2\theta - \cos^2\theta$

L.H.S. $\Rightarrow \sin^2\theta(1 + \sec^2\theta)$

$$\Rightarrow \quad \sin^2\theta\left(1 + \frac{1}{\cos^2\theta}\right) = \sin^2\theta + \tan^2\theta = (1 - \cos^2\theta) + (\sec^2\theta - 1)$$

$$= \sec^2\theta - \cos^2\theta$$

(iii) $\dfrac{1+\sin\theta}{1-\sin\theta} = (\sec\theta + \tan\theta)^2$

L.H.S. $\Rightarrow \dfrac{1+\sin\theta}{1-\sin\theta} \times \dfrac{1-\sin\theta}{1+\sin\theta} = \dfrac{(1+\sin\theta)^2}{1-\sin\theta} = \dfrac{(1+\sin\theta)^2}{\cos^2\theta}$

$$= \left(\frac{1+\sin\theta}{\cos\theta}\right) = \left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)$$

$$= (\sec\theta + \tan\theta)^2 = \text{R.H.S.}$$

(iv) $\sec\theta - \sin\theta = \dfrac{\tan^2\theta + \cos^2\theta}{\sec\theta + \sin\theta}$

$$\text{L.H.S.} \Rightarrow (\sec\theta - \sin\theta) \times \frac{\sec\theta + \sin\theta}{\sec\theta + \sin\theta}$$

$$= \frac{\sec^2\theta - \sin^2\theta}{\sec\theta + \sin\theta} = \frac{(\tan^2\theta + 1) - (1 - \cos^2\theta)}{\sec\theta + \sin\theta}$$

$$= \frac{\tan^2\theta + \cos^2\theta}{\sec\theta + \sin\theta} = \text{R.H.S.}$$

$$[\because \sec^2\theta = \tan^2\theta + 1, \sin^2\theta = 1 - \cos^2\theta]$$

(v) $\dfrac{\cos(A-B)\cos(A+B)}{\sin(A+B) + \sin(A-B)} = \tan\theta$

$$\text{L.H.S.} = \frac{\cos(A-B) - \cos(A+B)}{\sin(A+B) + \sin(A-B)}$$

$$= \frac{(\cos A \cdot \cos B + \sin A \sin A) - (\cos A \cos B - \sin A \sin A)}{(\sin A \cdot \cos B + \cos A \sin B) - (\sin A \cos B - \cos A \sin B)}$$

$$= \frac{\cos A \cdot \cos B + \sin A \sin B - \cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \cdot \sin B - \sin A \cos B - \cos A \sin B}$$

$$= \frac{2\sin A \sin B}{2s \sin A \cos B} = \frac{\sin B}{\cos B} = \tan B = \text{R.H.S.}$$

(vi) $\cos B - \cos A \cdot \cos(A-B) = \sin A \sin(A-B)$

$\text{L.H.S.} = \cos B - \cos A \cdot \cos(A-B)$

$\Rightarrow \quad \cos B - \cos A(\cos A \cdot \cos B + \sin A \sin B)$

$\Rightarrow \quad \cos B - \cos^2 A \cos B + \sin A \cos A \sin B$

$\Rightarrow \quad \cos B - (1 - \sin^2 A)\cos B - \sin A \cos A \sin B$

$\Rightarrow \quad \cos B - \cos B(1 - \sin^2 A) - \sin A \cos A \sin B$

$\cos B - \cos B + \cos B \sin^2 A - \sin A \cos A \sin B$

$\sin A(\cos B \sin A - \cos A \sin B] = \sin A \cdot \sin(A-B)$ R.H.S.

$$(vii)\ \tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\cdot\tan B\cdot\tan C}{1-\tan B\cdot\tan C\cdot\tan A-\tan A\cdot\tan B}$$

$$\text{L.H.S.}\Rightarrow \tan(A+B+C)=\frac{\tan(A+B)+\tan C}{1-\tan(A+B)\cdot\tan C}$$

$$\frac{\dfrac{\tan A+\tan B}{1-\tan A\cdot\tan B}+\tan C}{1-\dfrac{\tan A+\tan B}{1-\tan A\cdot\tan B}\cdot\tan C}$$

$$=\frac{\tan A+\tan B+\tan C-\tan A\cdot\tan B\cdot\tan C}{1-\tan A\cdot\tan B-\tan A\cdot\tan C-\tan B\tan C}=0$$

Since $A+B+C=180°\Rightarrow \tan(A+B+C)=0$

$$\frac{\tan A+\tan B+\tan C-\tan A\cdot\tan B\cdot\tan C}{1-\tan A\cdot\tan B-\tan A\cdot\tan C-\tan B\tan C}=0$$

$$\therefore \tan A+\tan B+\tan C=\tan A\cdot\tan B\cdot\tan C$$

$$(viii)\ \frac{1}{2}\left[\tan(x+h)+\tan(x-h)\right]-\tan x=\frac{\tan x\sin^2 h}{\cos^2 x-\sin^2 h}=0$$

$$\text{L.H.S.}\Rightarrow \frac{1}{2}\left[\tan(x+h)+\tan(x-h)-\tan x\right]\Rightarrow$$

$$\frac{1}{2}\left[\frac{\tan x+\tan \text{h}}{1-\tan x\cdot\tan \text{h}}+\frac{\tan x+\tan \text{h}}{1+\tan x\cdot\tan \text{h}}\right]-\tan x$$

$$=\frac{1}{2}\left[\frac{(1+\tan x\cdot\tan \text{h})(\tan x+\tan \text{h})+(1-\tan x\tan \text{h})(\tan x-\tan \text{h})}{1-\tan^2 x\tan^2 h}\right]-\tan x$$

$$\Rightarrow\frac{\tan x+\tan \text{h}+\tan^2 x\tan \text{h}+\tan x\tan^2 h+\tan x-\tan \text{h}-\tan^2 x\tan \text{h}+\tan^2 h\tan x}{2(1-\tan^2 x\tan^2 \text{h})}-\tan x$$

$$=\frac{1}{2}\left[\frac{2(\tan x+2\tan^2 \text{h}\tan x-\tan \text{h})}{1-\tan^2 x\tan^2 \text{h}}\right]-\tan x$$

$$=\frac{\tan x+\tan x\cdot\tan\text{h}^2 x}{1-\tan^2 x\tan^2 \text{h}}-\tan x$$

$$\Rightarrow\qquad \frac{\tan x(1+\tan^2 \text{h}x)}{1-\tan^2 x\tan^2 \text{h}}-\tan x$$

$$\Rightarrow\qquad \frac{\tan x\sec^2 \text{h}}{1-\tan^2 x\tan^2 \text{h}}-\tan x \qquad (\therefore 1+\tan^2\theta=\sec^2\theta)$$

$$\Rightarrow \quad \frac{\tan x \cdot \dfrac{1}{\cos^2 h}}{1 - \dfrac{\sin^2 x}{\cos^2 x} \cdot \dfrac{\sin^2 h}{\cos^2 h}} - \tan x$$

$$\Rightarrow \quad \frac{\tan x \cdot \dfrac{1}{\cos^2 h}}{\dfrac{\cos^2 x \cos^2 h - \sin^2 x \sin^2 h}{\cos^2 x \cos^2 h}} - \tan x$$

$$\Rightarrow \quad \frac{\left(\tan x \cdot \dfrac{1}{\cos^2 h}\right) \cdot \cos^2 x \cos^2 h}{\cos^2 x \cos^2 h - \sin^2 x \sin^2 h} - \tan x$$

$$= \frac{\tan x \cdot \cos^2 x}{\cos^2 x - \sin^2 h} - \tan x$$

$$\Rightarrow \quad \tan x \left[\frac{\cos^2 x}{\cos^2 x - \sin^2 h} - 1\right]$$

$$\tan x \left(\frac{\cos^2 x - (\cos^2 x - \sin^2 h}{\cos^2 x - \sin^2 h}\right)$$

$$\Rightarrow \tan x \left(\frac{\sinh^2}{\cos^2 x - \sin^2 h}\right) = \text{R.H.S.}$$

(*ix*) $\tan x = \sqrt{\dfrac{1 - \cos 2x}{1 + \cos 2x}}$

$$\text{R.H.S.} = \sqrt{\frac{\dfrac{1 - \cos 2x}{2}}{\dfrac{1 + \cos 2x}{2}}} = \sqrt{\frac{\sin^2 x}{\cos^2 x}} = \tan x = \text{L.H.S.}$$

(*x*) $\dfrac{\sin 4A + \sin 2A}{\cos 4A + \cos 2A + 1} = \dfrac{2 \sin 2A \cdot \cos 2A + \sin 2A}{\cos^2 2A - \sin^2 2A + \cos 2A + 1}$

$$\Rightarrow \quad \frac{\sin 2A\,(2 \cos 2A + 1)}{\cos^2 2A - 1(1 - \cos^2 2A) + \cos 2A + 1}$$

$$\Rightarrow \quad \frac{\sin 2A\,(2 \cos 2A + 1)}{\cos^2 2A - 1 + \cos^2 2A + \cos 2A + 1}$$

$$\frac{\sin 2A\,(2\cos 2A+1)}{\cos 2A\,(2\cos 2A+1)}=\tan 2A \Rightarrow \text{R.H.S.}$$

(*xi*) $\sin^4\theta+\cos^4\theta=\frac{1}{4}(\cos 4\theta+3)$

$$\text{R.H.S.}\Rightarrow \frac{1}{4}(\cos 4\theta+3)\Rightarrow \frac{1}{4}\left[\left(1-2\sin^2 2\theta\right)+3\right]$$

$$=\frac{1}{4}[(4-2\,(2\sin\theta-\cos\theta)^2]$$

$$=1-2\sin^2\theta\cos^2\theta=(\cos^2\theta+\sin^2\theta)-2\sin^2\theta\cos^2\theta$$

$$\Rightarrow \quad \cos^2\theta-\sin^2\theta\cos^2\theta+\sin^2\theta-\sin^2\theta\cos^2\theta$$

$$\cos^2\theta(1-\sin^2\theta)+\sin^2\theta(1-\cos^2\theta)=\cos^4\theta+\sin^4\theta=\text{R.H.S.}$$

(*xii*) $4\sin^3 A\cdot\cos 3A+4\cos^3 A\cdot\cos 3A\cdot\sin 3A=3\sin 4A$

$$\text{L.H.S.}\Rightarrow 4\sin^3 A\cos^3 A+4\cos^3 A\cdot\sin 3A$$

$$=\quad 4\sin 3\text{A}\,(\cos 2A\cos A-\sin 2A\cdot\sin A)+4\cos^3 A\,(\sin^2\cos A+\cos 2A\sin \text{A})$$

$$\Rightarrow \quad 4\sin^3 A\cos A\cos 2A-4\sin^4 A\sin 2A+4\cos^4 A\sin 2A+4\sin A\cos^3 A\cos 2\text{A}\ \ldots\ \ldots \quad (1)$$

Note $\sin 4A=2\sin 2A\cos 2A$

$\sin 2A=2\sin A\cos A$

$$\Rightarrow \quad 4\sin^3 A\cos A\cos 2A\Rightarrow 2\sin^2 A\cdot 2\sin A\cos A\cos^2 A$$

$$\Rightarrow \quad 2\sin 2A\sin^2 A\cos 2A \qquad \ldots(a)$$

$$4\sin A\cos^3 A\cos^2 A\Rightarrow 2\cos^2 A\,(2\cos A-\sin A)\cos 2A$$

$$\Rightarrow \quad 2\cos^2 A\sin 2A\cos 2A \qquad \ldots(b)$$

By substituting (*a*) and (*b*) in Equation (1),

$$2\sin^2 A\sin 2A\cos 2A+4\sin 2A\,(\cos^4 A-\sin^4 A)+2\cos^2 A\sin 4A$$

$$\therefore \sin^2 A\sin 4A+\cos^2 A\sin 4A+4\sin 2A\,(\cos^2 A-\sin^2 A)\,(\cos^2 A+\sin^2 A)$$

$$(\cos^2 A+\sin^2 A)$$

$$=\sin 4A\,(\sin^2 A+\cos^2 A)+4\sin 2A\cdot\cos 2A$$

$$=\sin 4A+2\sin 4A=3\sin 4A\Rightarrow \text{R.H.S.}$$

(*xiii*) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

$$\text{L.H.S.} \Rightarrow \tan(2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \cdot \tan A} = \frac{\dfrac{2\tan A}{1-\tan^2 A} + \tan A}{1 - \dfrac{2\tan A}{1-\tan^2 A}\cdot \tan A}$$

$\left[\text{As } \tan 2A = \dfrac{2\tan A}{1-\tan^2 A}\right]$

$$\Rightarrow \quad \frac{\dfrac{2\tan A + (1-\tan^2 A)\tan A}{1-\tan^2 A}}{1 - \dfrac{2\tan^2 A}{1-\tan^2 A}}$$

$$\Rightarrow \quad \frac{2\tan A + \tan A - \tan^3 A}{1 - \tan^2 A - 2\tan^2 A} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} = \text{R.H.S.}$$

(*xiv*) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

Let $\quad y = \pi - \cos^{-1} x \Rightarrow x = \cos(\pi - y)$

$\Rightarrow \quad x = -\cos y$

$\quad y = \cos^{-1}(-x)$

$\therefore \quad \cos^{-1}(-x) = \pi - y$

(*xv*) $\cos^{-1} x = \dfrac{\pi}{2} - \tan^{-1} x$

Let $\quad y = \dfrac{\pi}{2} - \tan^{-1} x \Rightarrow x = \tan\left(\dfrac{\pi}{2} - y\right)$

$\Rightarrow \quad x = \cot y \Rightarrow y = \cot^{-1} x$

$\therefore \quad \cot^{-1} x = \dfrac{\pi}{2} - \tan^{-1} x$

(*xvi*) (*a*) R.H.S. $= \cosh u.\cosh v + \sinh u \cdot \sinh v$

$$= \frac{e^u + e^{-u}}{2}\cdot\frac{e^v + e^{-v}}{2} + \frac{e^u - e^{-u}}{2}\cdot\frac{e^v - e^{-v}}{2} = \frac{e^{u+v} + e^{-(u+v)}}{2}$$

$= \cosh(u + v) =$ L.H.S.

(*b*) L.H.S. $= \cosh(u + (-v)) = \cosh u \cdot \cosh(-v) + \sinh u \cdot \sinh(-v)$

$= \cosh u \cdot \cosh v - \sinh u \cdot \sinh v =$ R.H.S.

$(xvii)$ $\cosh u \cdot \sinh v = 1/2\,[\sinh(u+v) - \sinh(u-v)]$

$$\text{R.H.S} = 1/2\,[(\sinh u \cdot \cosh v + \cosh u \cdot \sinh v) - (\sinh u \cdot \cosh v - \cosh u \cdot \sinh v)]$$
$$= 1/2\,[\sinh u \cdot \cosh v + \cosh u \cdot \sinh v - \sinh u \cdot \cosh v + \cosh u - \sin v]$$
$$= 1/2\,[2 \cosh u \cdot \sinh v]$$
$$= \cosh u \cdot \sinh v = \text{L.H.S.}$$

$(xviii)$ $\sinh u \cdot \sinh v = 1/2\,(\cosh(u+v) - \cosh(u-v)$

$$\text{R.H.S.} = 1/2\,(\cosh(u+v) - \cosh(u-v)]$$
$$= 1/2(\cosh u \cdot \cosh v + \sinh v + \sinh v) - (\cosh u, \cosh v - \sinh u, \sinh v)]$$
$$\therefore 1/2\,[\cosh u \cdot \cosh v + \sinh u + \sinh v - \cosh u \cdot \text{cosnh}\, v + \sinh v \cdot \sinh v]$$
$$= 1/2\,[2 \sinh u \cdot \sinh v]$$
$$= \sinh u \cdot \sinh v = \text{L.H.S.}$$

(xix) $\cosh 3u = \cosh u + 4 \sinh^2 \cdot \cosh u = 4 \cosh 3u - 3 \cosh u$

$$\text{L.H.S.} = \cosh 3u = \cos h\,(2u+u)$$
$$= \cosh 2u \cdot \cosh u + \sin 2u \cdot \sinh 2u$$
$$= \cosh^2 u + \sin h^2 u$$
$$= (\cosh^2 u + \sinh^2 u) \cdot \cosh u + 2\,(\sinh u \cdot \cosh u) \sinh u$$
$$= \cosh^3 u + \sinh^2 u \cdot \cosh u + 2 \sinh^2 u \cdot \cosh u$$
$$= \cosh^3 u + 3 \sinh^2 u \cdot \cosh u$$
$$= \cosh^3 u + 3\,(\cosh^2 u - 1) \cosh u$$
$$= \cosh^3 u + 3 \cosh^3 u - 3 \cosh u$$
$$= 4 \cosh^3 u - 3 \cosh u = \text{R.H.S.}$$

(xx) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

$$\text{L.H.S.} = (\cosh x + \sinh x)^n$$
$$= (ex)^n$$
$$= e^{nx}$$

Let $nx = y$

$$e^{nx} = e^y$$
$$= \cosh y + \sinh y$$
$$= \cosh nx + \sinh nx = \text{R.H.S.}$$

PROBLEM 2.10

If $u = \dfrac{1+\sin\theta}{\cos\theta}$, prove that $\dfrac{1}{u} = \dfrac{1-\sin\theta}{\cos\theta}$ and deduce the formula for $\sin\theta$, $\cos\theta$, and $\tan\theta$ in terms of u.

Solution:

$$u = \frac{1+\sin\theta}{\cos\theta} \text{ prove that } \frac{1}{u} = \frac{1-\sin\theta}{\cos\theta}$$

$$u = \frac{1+\sin\theta}{\cos\theta} \Rightarrow \frac{1}{u} = \frac{\cos\theta}{1+\sin\theta} \times \frac{1-\sin\theta}{1-\sin\theta} = \frac{\cos(1-\sin\theta)}{1-\sin^2\theta}$$

$$= \frac{1-\sin\theta}{\cos\theta}$$

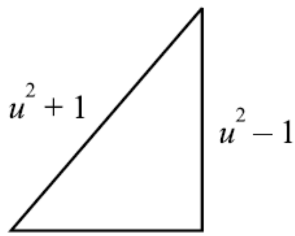
$$u^2 = \frac{(1+\sin\theta)^2}{\cos^2\theta} \Rightarrow u^2 = \frac{(1+\sin\theta)^2}{1-\sin^2\theta} = \frac{(1+\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)}$$

$$u^2 = \frac{1+\sin\theta}{1-\sin\theta} \Rightarrow (1-\sin\theta)u^2 = 1+\sin\theta$$

$$u^2 - u^2\sin\theta = 1+\sin\theta = \sin\theta$$

$$u^2 - 1 = \sin\theta + u^2\sin\theta$$

$$u^2 - 1 = \sin\theta\left(u^2+1\right)$$



$$\therefore \quad \sin\theta = \frac{(u^2-1)}{(u^2+1)}$$

$$\therefore \quad \cos\theta = \frac{2u}{u^2+1}$$

$$\therefore \quad \tan\theta = \frac{(u^2-1)}{(u^2+1)}$$

PROBLEM 2.11

If $\sin(x+\alpha) = 2\cos(x-\alpha)$, prove that $\tan x = \dfrac{2-\tan\alpha}{1-2\tan\alpha}$.

Solution: $\sin x \cdot \cos\alpha + \cos\alpha \sin\alpha = 2(\cos\alpha \cos\alpha + \sin\alpha \sin\alpha)$

Dividing throughout by $\cos x \cos\alpha$, we obtain

$$\Rightarrow \quad \tan x + \tan\alpha = 2 + 2\tan x \cdot \tan\alpha$$

$$\therefore \quad \tan x + \tan\alpha = 2 + 2\tan x \cdot \tan\alpha$$

$$\therefore \quad \tan x = \frac{2-\tan x}{1-2\tan\alpha}$$

PROBLEM 2.12

If $\sin(x-\alpha)=\cos(x+\alpha)$ prove that $\tan x = 1$.

Solution: $(\sin x\cdot\cos\alpha-\cos\alpha\,\sin\alpha=\cos x\cos\alpha-\sin x\cdot\sin\alpha)$

$$\Rightarrow \tan x\cdot\cos\alpha-\sin\alpha=\cos\alpha-\tan\alpha\; x$$

$$\therefore \tan x\cdot\cos\alpha+\tan\alpha\,\sin\alpha=\cos\alpha+\sin\alpha$$

$$\therefore \tan x\cdot\cos\alpha+\tan\alpha\,\sin\alpha=\cos\alpha+\sin\alpha$$

$$\therefore \tan\alpha\,(\cos\alpha+\sin\alpha)=\cos\alpha+\sin\alpha$$

$$\therefore \tan x=\frac{\cos\alpha+\sin\alpha}{\cos\alpha+\sin\alpha}\Rightarrow \tan\alpha=1$$

PROBLEM 2.13

If $x=\cos\theta+\cos 2\theta$ and $y=\sin\theta+\sin 2\theta$, show that
(*i*) $x^2-y^2=\cos 2\theta+2\cos 3\theta+\cos 4\theta$

(*ii*) $2xy=\sin 2\theta+2\sin 3\theta+\sin 4\theta$

Solution: (*i*) $x^2-y^2=\cos^2\theta+2\cos^3\theta+\cos^4\theta$

$$\text{L.H.S.}=x^2-y^2=(\cos\theta+\cos 2\theta)^2-(\sin\theta+\sin 2\theta)^2$$

$$=\cos^2\theta+2\cos\theta\cos 2\theta+\cos^2 2\theta-\sin^2\theta-2\sin\theta\sin^2\theta-\sin^2 2\theta.$$

$$=(\cos^2\theta-\sin^2\theta)+(\cos^2 2\theta-\sin^2 2\theta)$$

$$=\sin 2\theta+2\sin 3\theta+\sin 4\theta=\text{R.H.S.}$$

(*ii*) $2\,xy=\sin 2\theta+2\sin 3\theta+\sin 4\theta$

$$\text{L.H.S.}=2\,(\cos\theta+\cos 2\theta)\,(\sin\theta+\sin 2\theta)=2\sin\theta\cos\theta+2\cos\theta\sin 2\theta+2\sin\theta\cos 2\theta+2\sin 2\theta\cos 2\theta$$

$$=\sin 2\theta+2\sin 3\theta+\sin 4\theta=\text{R.H.S.}$$

PROBLEM 2.14

If $\cos 2A\cdot\cos 2B=\cos 2\theta$, prove that

$\sin^2 A\cdot\cos^2 B+\cos^2 A\cdot\sin^2 B=\sin^2\theta$

Solution: L.H.S. $=\sin^2 A\cos^2 B+\cos^2 A\sin^2 B$

$$=\frac{1-\cos 2A}{2}\cdot\frac{1+\cos 2B}{2}+\frac{1+\cos 2A}{2}\cdot\frac{1-\cos 2B}{2}$$

$$= \frac{1}{4}(1 + \cos 2A - \cos 2A - \cos 2A \cos 2B) + \frac{1}{4}(1 + \cos 2A - \cos 2B - \cos 2A \cos 2B)$$
$$= \frac{1}{4}(2 - 2 \cos 2A \cos 2B) + \frac{1}{2}(1 - \cos 2A - \cos 2B)$$
$$= \frac{1}{2}(1 - \cos 2\theta) = \sin^2 \theta = \text{R.H.S.}$$

PROBLEM 2.15

If $S = \sin\theta$ and $C = \cos\theta$, simplify:

(*i*) $\dfrac{S \cdot C}{\sqrt{1 - S^2}}$

(*ii*) $\dfrac{S \cdot \sqrt{1 - S^2}}{C \cdot \sqrt{1 - C^2}}$

(*iii*) $\dfrac{C}{S} + \dfrac{S}{C}$

Solution: (*i*) $\dfrac{S \cdot C}{\sqrt{1 - S^2}} = \dfrac{\sin\theta\cos\theta}{\sqrt{1 - \sin^2\theta}} = \dfrac{\sin\theta\cos\theta}{\sqrt{\cos^2\theta}}$

$$= \frac{\sin\theta\cos\theta}{\cos\theta} = \sin\theta$$

(*ii*) $\dfrac{S.\sqrt{1 - S^2}}{C\sqrt{1 - C^2}} \Rightarrow \dfrac{\sin\theta\sqrt{1 - \sin^2\theta}}{\cos\theta\sqrt{1 - \cos^2\theta}} = \dfrac{\sin\theta\cos\theta}{\cos\theta\sin\theta} = 1$

(*iii*) $\dfrac{\text{C}}{\text{S}} + \dfrac{\text{S}}{\text{C}} \rightarrow \dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta} = \dfrac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} = \dfrac{1}{\cos\theta} \cdot \dfrac{1}{\sin\theta} = \sec\theta\cos\theta$

PROBLEM 2.16

Eliminate θ from the following equations:

(*i*) $x = a \cdot \text{cosec}\,\theta$ **and** $y = b \cdot \sec\theta$

(*ii*) $x = \sin\theta + \cos\theta$ **and** $y = \sin\theta - \cos\theta$

(*iii*) $x \sin\theta + \tan\theta$ **and** $y = \sin\theta - \tan\theta$

(*iv*) $x = \tan\theta$ **and** $y = \tan 2\theta$

Solution: (i) $x = a \operatorname{cosec}\theta$ and $= b \cdot \sec\theta$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\therefore \quad x = a\left(\frac{1}{\sin\theta}\right) \Rightarrow x = \frac{a}{\sin\theta}$$

$$x\sin\theta = a$$

$$\Rightarrow \quad y = b\sec\theta \therefore \ \sec\theta = \frac{1}{\cos\theta}$$

$$y = b\left(\frac{1}{\cos\theta}\right) \Rightarrow y\cos\theta = b \therefore \cos\theta = \frac{b}{y}$$

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \therefore \left(\frac{a}{x}\right)^2 + \left(\frac{b}{y}\right)^2 = 1$$

$$\Rightarrow \quad \frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$$

(ii) $x = \sin\theta + \cos\theta, \quad y = \sin\theta - \cos\theta$

After adding, we obtain

$$x = \sin\theta + \cos\theta$$
$$y = \sin\theta - \cos\theta$$
$$x + y = 2\sin\theta$$
$$\therefore \sin\theta = \frac{x+y}{2}$$

After subtracting, we obtain

$$x = \sin\theta + \cos\theta$$
$$-y = \sin\theta + \cos\theta$$
$$x - y = 2\cos\theta$$
$$\therefore \cos\theta = \frac{x-y}{2}$$

$$\sin^2\theta + \cos^2\theta = 1 \rightarrow \left(\frac{x+y}{2}\right) + \left(\frac{x-y}{2}\right)^2 = 1$$

$$\frac{x^2 + 2xy + y^2}{4} + \frac{x^2 - 2xy + y^2}{4} = 1 \Rightarrow \frac{2\left(x^2 + y^2\right)}{4} = 1 \Rightarrow x^2 + y^2 = 2$$

(iii) $x = \sin\theta + \tan\theta$

$y = \sin\theta - \tan\theta$

On adding, we obtain

$$x + y = 2\sin\theta \Rightarrow \sin\theta = \frac{x+y}{2}$$

On subtracting, we obtain

$$x - y = 2\tan\theta$$

$$\Rightarrow \quad \tan\theta = \frac{x-y}{2}$$

$$\Rightarrow \quad \frac{\sin\theta}{\cos\theta} = \frac{x-y}{2}$$

$$\therefore \quad \frac{\frac{x+y}{2}}{\cos\theta} = \frac{x-y}{2}$$

$$\Rightarrow \quad \cos\theta = \frac{x+y}{x-y}$$

Since,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\frac{x+y}{2}\right)^2 + \left(\frac{x+y}{x-y}\right)^2 = 1$$

$$\frac{(x+y)^2}{4} + \frac{(x+y)^2}{(x-y)^2} = 1$$

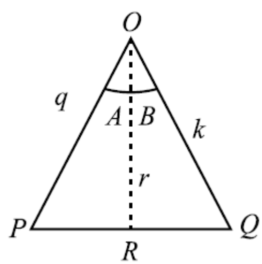
$$\Rightarrow \quad \frac{4}{(x+y)^2} - \frac{4}{(x-y)^2} = 1$$

(*iv*) $x = \tan\theta$ and $y = \tan 2\theta$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \Rightarrow y = \frac{2x}{1-x^2}$$

PROBLEM 2.17

In the acute-angled triangle OPQ, the altitude OR makes angles A and B with OP with OQ. Show by means of areas that if $OP = q, OQ = k, OR = r : k.q.\sin(A+B) = q.r.\sin A + k.r.\sin B$.



Solution: $\cos A = \dfrac{r}{q} \qquad \cos B = \dfrac{r}{k}$

$$kq \sin(A+B) = qr \sin A + kr \sin B$$

$$\text{L.H.S.} \quad kq \sin(A+B) = kq(\sin A \cos B + \cos A \sin B)$$

$$= kq\left(\sin A \frac{r}{k} + \frac{r}{q}\sin B\right)$$

$$= \frac{kq}{k}\sin A + \frac{kq}{k}\sin B$$

$$= qr \sin A + kr \sin B = \text{R.H.S.}$$

PROBLEM 2.18

Given that $\alpha = \sin^{-1}\frac{1}{2}$, find $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, and $\operatorname{cosec}\alpha$.

Solution: Since,

$$\alpha = \sin^{-1}\frac{1}{2} \Rightarrow \alpha = 30°$$

$$\cos 30 = \frac{\sqrt{3}}{2}, \tan\alpha = \frac{1}{\sqrt{3}}$$

$$\sec\alpha = \frac{2}{\sqrt{3}}, \operatorname{cosec}\alpha = 2$$

PROBLEM 2.19

Evaluate the following expressions:

(*a*) $\sin(\cos^{-1}\frac{1}{\sqrt{2}})$

(*b*) $\operatorname{cosec}(\sec^{-1} 2)$

(*c*) $\cot(\cos^{-1} 0)$

(*d*) $\sin^{-1} 1 - \sin^{-1}(-1)$

(*e*) $\cos(\sin^{-1} 0.8)$

(*f*) $\cos^{-1}\left(-\sin\frac{\pi}{6}\right)$

Solution:

$$(a)\ \cos^{-1}\frac{1}{\sqrt{2}} = 45° = \frac{\pi}{2}$$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$(b)\ \operatorname{cosec}(\sec^{-1} 2) = \operatorname{cosec}\frac{\pi}{3} = \frac{2}{\sqrt{3}}$$

$$(c)\ \cot(\cos^{-1} 0) = \cot\frac{\pi}{2} = 0$$

$$(d)\ \sin^{-1}1 - \sin^{-1}(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

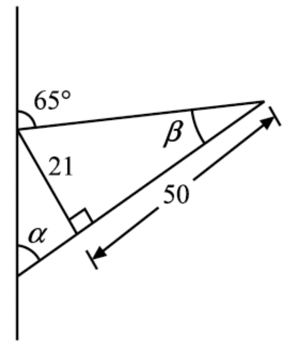
$$(e)\ \cos(\sin^{-1}(0.8)) = \frac{6}{10} = 0.6$$

$$(f)\ \cos^{-1}\left(-\sin\frac{\pi}{6}\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{2}{3}$$

PROBLEM 2.20

Find the angle α in the graph (Hint: $\alpha + \beta = 65°$).

Solution:



$$\alpha + \beta = 65°$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$$

$$\tan\beta = \frac{21}{50}$$

$$\therefore \quad \tan(\alpha + \beta) \Rightarrow \tan 65 = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$$

$$\tan 65° = \frac{\tan\alpha + \frac{21}{50}}{1-\tan\alpha\cdot\frac{21}{50}} \quad \frac{21}{50} = 0.42$$

$$\tan 65\,(1-\tan\alpha\,(0.42)) = \tan\alpha + 0.42$$

$$\tan 65 - 0.42 = \tan\alpha\ \tan 65° - \tan\alpha - 0.42$$

$$\tan 65° - 0.42 = 0.42\tan\alpha\ \tan 65° + \tan\alpha$$

$$\tan 65° - 0.42 = \tan\alpha\,(0.42\tan 65° + 1)$$

$$\therefore \quad \tan\alpha = \frac{\tan 65 - 0.42}{0.42\tan 65 + 1} \Rightarrow \tan\alpha = \frac{2.14-0.42}{0.42(2.14)+1}$$

$$\therefore \quad \tan\alpha = \frac{1.724}{1.8485} \Rightarrow \tan\alpha = 0.4079$$

$$\therefore \quad \tan - 0.907$$

$$\therefore \quad \alpha = 42.2°$$

[**Ans.** 42.2]

PROBLEM 2.21

Let sech $u = 3/5$; determine the values of the remaining five hyperbolic functions.

Solution: $\cosh u = \frac{1}{\sec hu} \rightarrow \cosh u = \frac{5}{3}$

$$\tanh^2 u + \text{sech}^2 u = 1 \rightarrow \tanh^2 u + \frac{9}{25} = 1 \Rightarrow \tanh u = \mp\frac{4}{5}$$

$$\coth u = \frac{1}{\tanh u} \rightarrow \coth u = \mp\frac{5}{4}$$

$$\tanh u = \frac{\sinh u}{\cosh u} \Rightarrow \mp\frac{4}{5} = \frac{\sinh u}{5/3} \Rightarrow \sinh u \mp \frac{4}{3}$$

$$\text{csch}\ u = \frac{1}{\sinh u} \Rightarrow \sec h\, u = \mp\frac{3}{4}$$

PROBLEM 2.22

Rewrite the following expressions in terms of exponentials; write the final result as simply as you can:

(*a*) $\sinh(2\cdot\ln x)$

(*b*) $\dfrac{1}{\cosh x - \sinh x}$

(*c*) **cosh 3*x* – sinh 3*x***
(*d*) **ln (cosh *x* + sinh *x*) + ln (cosh *x* – sinh *x*)**

Solution:

(*a*) $\sin h\,(2\ln x) = \dfrac{e^{2\ln x} - e^{-2\ln x}}{2} = \dfrac{e^{\ln x^2} - e^{-\ln x^2}}{2} = \dfrac{x^2 - \dfrac{1}{x^2}}{2} \times x^2 = \dfrac{x^4 - 1}{2x^2}$

(*b*) $\dfrac{1}{\cosh x - \sinh x} \Rightarrow \dfrac{1}{\dfrac{e^x + e^{-x}}{2} - \dfrac{e^x - e^{-x}}{2}} = \dfrac{1}{\dfrac{e^{3x} + e^{-3x} - e^{3x} - e^{-3x}}{2}} = \dfrac{1}{\dfrac{2e^{-3x}}{2}} = \dfrac{1}{e^{-x}} = e^x$

(*c*) $\cosh 3x - \sinh 3x \Rightarrow \dfrac{e^{3x} + e^{-3x}}{2} - \dfrac{e^{3x-} - e^{-3x}}{2}$

$$= \frac{e^{3x} + e^{-3x} - e^{3x} + e^{-3x}}{2} = \frac{2e^{-3x}}{2} = e^{-3x}$$

(*d*) $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$

$\ln[\cosh x + \sinh x) + (\cosh x - \sinh x)]$

$\ln[(\cosh^2 x - \sinh^2 x)] = \ln 1 = 0$

PROBLEM 2.23

Solve the equation for *x*; tanh *x* = 3 / 5.

Solution: $\tanh x = \dfrac{3}{5} \Rightarrow \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{3}{5}$

$$5\,(e^x - e^{-x}) = 3\,(e^x + e^{-x})$$

$$5e^x - 5e^{-x} = 3e^x + 3e^{-x}$$

$$5e^x - 3e^x = 3e^{-x} + 5e^{-x}$$

$$2e^x = 8e^{-x} \div 2$$

$$e^x = 4e^{-x}$$

$$\frac{e^x}{e^{-x}} = 4 \Rightarrow e^x \cdot e^x = 4 \Rightarrow e^{2x} = 4$$

$$2x = \ln 4$$

$$x = \frac{\ln 4}{2}$$

PROBLEM 2.24

Show that the distance r from the origin O to the point $P(\cosh u, \sinh u)$ on the hyperbola $x^2 - y^2 = 1$ is $r = \sqrt{\cosh 2u}$.

Solution: $(0, 0)\,(\cosh u, \sinh u)$

$$r = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

$$r = \sqrt{(\cosh u - 0)^2 + (\sinh u - 0)^2}$$

$$= \sqrt{\cosh^2 u + \sinh^2 u} = \sqrt{\cosh 2u}$$

PROBLEM 2.25

θ lies in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\sinh x = \tan\theta$. Show that $\cosh x = \sec\theta$, $\tanh x = \sin\theta$, $\coth x = \operatorname{cosec}\theta$, $\operatorname{cosech} x = \cot\theta$, and $\operatorname{sech} x = \cos\theta$.

Solution: (a) since $\sinh x = \tan\theta$

$$\sinh^2 x = \tan^2\theta$$

$$\cosh^2 x - 1 = \tan^2\theta \qquad (\because \sinh^2 x = \cosh^2 x - 1)$$

$$\cosh^2 x = \tan^2\theta + 1$$

$$\cosh^2 x = \sec^2\theta \qquad (\because 1 + \tan^2\theta = \sec^2\theta)$$

$$\Rightarrow \qquad \cosh x = \sec\theta$$

$$(b)\ \tanh x = \frac{\sinh x}{\cosh x} = \frac{\tan\theta}{\sec\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} = \frac{\sin\theta}{\cos\theta} = \sin\theta$$

$$(c)\ \coth x = \frac{1}{\tanh x} = \frac{1}{\sin\theta} = \operatorname{cosec}\theta$$

$$(d)\ \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{1}{\tan\theta} = \cot\theta$$

$$(e)\ \operatorname{sech} x = \frac{1}{\cosh x} = \frac{1}{\sec\theta} = \frac{1}{\frac{1}{\cos\theta}} = \cos\theta$$

PROBLEM 2.26

Derive the formula: $\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x};|x|<1$

Solution: $y = \tanh^{-1} x \Rightarrow x = \tanh y$

$$\therefore \quad x = \frac{e^y - e^{-y}}{e^y + e^{-y}} \times \frac{e^y}{e^y}$$

$$x = \frac{e^{2y}-1}{e^{2y}+1} \Rightarrow xe^{2y} + x = e^{2y} - 1$$

$$\therefore \quad e^{2y} = \frac{1+x}{1-x}$$

$$2y = \ln\frac{1+x}{1-x}$$

$$y = \frac{1}{2}\ln\frac{1+x}{1-x}$$

$$\therefore \quad \tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$

PROBLEM 2.27

Find : $\lim_{x\to\infty}\left[\cosh^{-1} x - \ln x\right]$

Solution: $\lim_{x\to\infty}(\cosh^{-1} x - \ln x) \Rightarrow \lim_{x\to\infty}\left(\ln(x+\sqrt{x^2-1} - \ln x)\right)$

$$= \lim_{x\to\infty} \ln\left(\frac{x+\sqrt{x^2-1}}{x}\right)$$

$$= \lim_{x\to\infty}\left(1+\sqrt{\frac{x^2-1}{x^2}}\right)$$

$$= \ln \lim_{x\to\infty}\left(1+\sqrt{\frac{x^2-1}{x^2}}\right)$$

$$= \ln\left(1+\sqrt{1-0}\right) = \ln 2$$

CHAPTER 3

DERIVATIVES

PROBLEMS

PROBLEM 3.1

Find $\dfrac{dy}{dx}$ for the following functions:

1. $y=(x-3)(1-x)$ [**Ans.** $4-2x$]

2. $y=\dfrac{ax+b}{x}$ [**Ans.** $a+bx^{-1}$]

3. $y=\dfrac{3x+4}{2x+3}$ [**Ans.** $\dfrac{1}{(2x+3)^2}$]

4. $y=3x^3-2\sqrt{x}+\dfrac{5}{x^2}$ [**Ans.** $9x^2-\dfrac{1}{\sqrt{x}}-\dfrac{10}{x^3}$]

5. $y=\left(\sqrt{x^3}-\dfrac{1}{\sqrt{x^3}}\right)$ [**Ans.** $\dfrac{3(x^6-1)}{x^4}$]

6. $y=(2x-1)^2(3x+2)^3+\dfrac{1}{(x-2)^2}$ [**Ans.** $(2x-1)(3x+2)^2(30x-1)-\dfrac{2}{(x-2)^3}$]

7. $y=\ln(\ln x)$ [**Ans.** $\dfrac{1}{x.\ln x}$]

8. $y=\ln(\cos x)$ [**Ans.** $-\tan x$]

9. $y = \sin x^3$ [**Ans.** $3x^2 \cdot \cos x^3$]

10. $y = \cos^{-3}\left(5x^2 + 2\right)$ [**Ans.** $+30x\dfrac{\sin(5x^2+2)}{\cos^4(5x^2+2)}$]

11. $y = \tan x \cdot \sin x$ [**Ans.** $\sin x + \tan x \cdot \sec x$]

12. $y = \tan(\sec x)$ [**Ans.** $\sec^2(\sec x) \cdot \sec x \cdot \tan x$]

13. $y = \cot^3\left(\dfrac{x+1}{x-1}\right)$ [**Ans.** $\dfrac{6}{(x-1)^2} \cdot \cot^2\left(\dfrac{x+1}{x-1}\right) \cdot \operatorname{cosec}^2\left(\dfrac{x+1}{x-1}\right)$]

14. $y = \dfrac{\cos x}{x}$ [**Ans.** $\dfrac{x \cdot \sin x + \cos x}{x^2}$]

15. $y = \tan^{1/2}\sqrt{2x+7}$ [**Ans.** $\dfrac{\sec^2\sqrt{2x+7}}{2(\tan^{1/2}\sqrt{2x+7})(\sqrt[2]{2x+7})}$]

16. $y = x^2 . \sin x$ [**Ans.** $x^2 \cdot \cos x + 2x \cdot \sin x$]

17. $y = \operatorname{cosec}^{-\frac{2}{3}}\sqrt{5x}$ [**Ans.** $\dfrac{5}{3\sqrt{5x}} \cdot \dfrac{\cot\sqrt{5x}}{\operatorname{cosec}^{\frac{2}{3}}\sqrt{5x}}$]

18. $y = x[\sin(\ln x) + \cos(\ln x)]$ [**Ans.** $2 \cdot \cos(\ln x)$]

19. $y = \sin^{-1}(5x^2)$ [**Ans.** $\dfrac{10x}{\sqrt{1-25x^4}}$]

20. $y = \cot^{-1}\left(\dfrac{1+x}{1-x}\right)$ [**Ans.** $-\dfrac{1}{1+x^2}$]

21. $y = \tan^{-1}\sqrt{4x^3-2}$ [**Ans.** $\dfrac{6x^2}{(4x^3-1)\sqrt{4x^3-2}}$]

22. $y = \sec^{-1}(3x^2+1)^3$ [**Ans.** $\dfrac{18x}{\left|3x^2+1\right|\sqrt{(3x^2+1)^6-1}}$]

23. $y = \sin^{-1}\dfrac{x^2}{2-x} + x^2\sec^{-1}\dfrac{x}{2}$ [**Ans.** $\dfrac{4x-x^2}{(2-x)\sqrt{(2-x)^2-x^4}} + \dfrac{2x}{\sqrt{x^2-4}} + 2x \cdot \sec^{-1}\dfrac{x}{2}$]

24. $y = \sin^{-1}2x.\cos^{-1}2x$ [**Ans.** $\dfrac{2(\cos^{-1}2x - \sin^{-1}2x)}{\sqrt{1-4x^2}}$]

25. $y=\sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

[**Ans.** $\dfrac{1}{3}\sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}\cdot\left[\dfrac{1}{x}+\dfrac{1}{x+1}+\dfrac{1}{x-2}-\dfrac{2x}{x^2+1}-\dfrac{2}{2x+3}\right]$]

26. $y^{\frac{4}{5}}=\dfrac{\sqrt{\sin x\cdot\cos x}}{1+2\cdot\ln x}$ [**Ans.** $\dfrac{dy}{dx}=\dfrac{5}{8}y\left(\dfrac{1}{2}\cot x-\dfrac{1}{2}\tan x-\dfrac{2}{x(1+2\ln x)}\right)$]

27. $\sqrt{y}=\dfrac{x^5\cdot\tan^{-1}x}{(3-2x)\sqrt[3]{x}}$ [**Ans.** $2y\left(\dfrac{1}{(1+x^2)\tan^{-1}x}+\dfrac{2}{3-2x}+\dfrac{14}{3x}\right)$]

28. $y=\sec^{-1}(e^{2x})$ [**Ans.** $\dfrac{2}{\sqrt{e^{4x}-1}}$]

29. $y=(\cos x)^{\sqrt{x}}$ [**Ans.** $\dfrac{y}{2\sqrt{x}}(\ln\cos x-2x\cdot\tan x)$]

30. $y=(\sin x)\tan x$ [**Ans.** $y\,(1+\sec^2 x\cdot\ln\sin x)$]

31. $y=\sqrt{2x^2+\cosh^2(5x)}$ [**Ans.** $\dfrac{2x+5\cosh(5x)\cdot\sinh(5x)}{\sqrt{2x^2+\cosh^2(5x)}}$]

32. $y=\sinh(\cos 2x)$ [**Ans.** $2\sin 2x\cdot\cosh(\cos 2x)$]

33. $y=\operatorname{csc}h\dfrac{1}{x}$ [**Ans.** $\dfrac{1}{x^2}\cdot\operatorname{csch}\dfrac{1}{x}\cdot\coth\dfrac{1}{x}$]

34. $y=x^2\cdot\tanh^2\sqrt{x}$ [**Ans.** $x\cdot\tanh\sqrt{x}(x\operatorname{sech}^2\sqrt{x}+2\tanh\sqrt{x})$]

35. $y=\ln\dfrac{\sin x\cdot\cos x+\tan^3 x}{\sqrt{x}}$ [**Ans.** $\dfrac{\cos^2 x-\sin^2 x+3\tan^2 x\cdot\sec^2 x}{\sin x\cdot\cos x+\tan^3 x}-\dfrac{1}{2x}$]

36. $y=\log_4\sin x$ [**Ans.** $\dfrac{\cot x}{\ln 4}$]

37. $y=e^{x^2-5x}\cdot(2x-5e^{5u})$ [**Ans.** $(2x-5e^{5x})e^{(x2-3e5x)}$]

38. $y=e^{x^2}+\tan x$ [**Ans.** $(x^2\sec^2 x+2x\tan x)e^{x2\tan x})$]

39. $y = 7^{\csc\sqrt{2x+3}}$ [**Ans.** $-7^{\text{cosec}\sqrt{2x+3}}\dfrac{\text{cosec}\sqrt{2x+3}}{\sqrt{2x+3}}\cdot\dfrac{\cot\sqrt{2x+3}}{1}$]

40. $y = \left[\ln(x^2+2)^2\right]\cos x$ [**Ans.** $\dfrac{4x\cdot\cos x}{x^2+2} - 2\ln(x^2+2)\sin x)$]

41. $y = \sinh^{-1}(\tan x)$ [**Ans.** $|\sec x|$]

42. $y = \sqrt{1+(\ln x)^2}$ [**Ans.** $\dfrac{\ln x}{x\sqrt{1+\ln x^2}}$]

43. $y = \dfrac{e^x}{\ln x}$ [**Ans.** $\dfrac{e^x(x\ln x - 1)}{x(\ln x)^2}$]

44. $y = x^3\log_2(3-2x)$ [**Ans.** $3x^2\log_2(3-2) - \dfrac{2x^3}{(3-2x)\ln 2}$]

45. $y = 2\cosh^{-1}\dfrac{x}{2} + \dfrac{x}{2}\sqrt{x^2-4}$ [**Ans.** $\dfrac{x^2}{\sqrt{x^2-4}} + \dfrac{x^2}{2\sqrt{x^2+4}} + \dfrac{\sqrt{x^2+4}}{2}$]

Solution:

1. $y = (x-3)(1-x)$

$$\frac{dy}{dx} = (x-3)(-1) + (1-x)(1)$$
$$= -x+3+1-x = 4-2x$$

2. $y = \dfrac{ax+b}{x} = a + \dfrac{b}{x} = a + bx^{-1}$

3. $\dfrac{dy}{dx} = 0 + b(-1x^{-2})$

$$y = \frac{3x+4}{2x+3} = \frac{(2x+3)-(3x+4)(2)}{(2x+3)^2}$$
$$= \frac{6x+9-6x-8}{(2x+3)^2} = \frac{1}{(2x+3)^2}$$

4. $y = 3x^3 - 2\sqrt{x} + \dfrac{5}{x^2}$

$$y = 3x^3 - 2(x)^{\frac{1}{2}} + 5(x)^{-2}$$

$$\therefore \quad \frac{dy}{dx} = 9x^2 - 2 \times \frac{1}{2}(x)^{\frac{-1}{2}} + 5(-2x^{-2-1})$$

$$= 9x^2 - \frac{1}{\sqrt{x}} - \frac{10}{x^3}$$

5. $y = \left(\sqrt{x^3} - \dfrac{1}{\sqrt{x^3}}\right)^2$

$$y = \left(x^3 - 2\sqrt{x^3}\,\frac{1}{\sqrt{x^3}} + \frac{1}{x^3}\right)$$

$$y = x^3 - 2 + x^{-3} \quad \therefore \quad \frac{dy}{dx} = 3x^2 - 0 - 3x^{-4}$$

$$= 3x^2 - \frac{3}{x^4}$$

$$= \frac{3x^6 - 3}{x^4} = \frac{3(x^6 - 1)}{x^4}$$

6. $y = (2x-1)^2(3x+2)^3 + \dfrac{1}{(x-2)^2} = (2x-1)^2(3x+2)^3 + (x-2)^{-2}$

$$\therefore \quad \frac{dy}{dx} = (2x-1)^2\,3(3x+2)^2\,(3) + (3x+2)^3\,2(2x-1)(2) - 2(x-2)^{-3}$$

$$= 9(2x-1)^2(3x+2)^2 + 4(2x-1)(3x+2)^3 - \frac{2}{(x-2)^3}$$

$$= (2x-1)(3x+2)^2[18x - 9 + 12x + 8] - \frac{2}{(x-2)^3}$$

$$= (2x-1)(3x+2)^2(30x-1) - \frac{2}{(x-2)^3}$$

7. $y = \ln(\ln x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{(\ln x)} \cdot \frac{1}{x} = \frac{1}{x \ln x} \qquad \left(\because \ \frac{d}{dx}\ln x = \frac{1}{x}\right)$$

8. $y = \ln(\cos x)$

$$\frac{dy}{dx} = \frac{1}{\cos x} \cdot (-\sin x) = \frac{-\sin x}{\cos x} = -\tan x$$

9. $y = \sin x^3 \Rightarrow$

$$\frac{dy}{dx} = \cos x^3 \times 3x^2 = 3x^2 \cos x^3$$

10. $y = \cos^{-3}(5x^2 + 2)$

$$y = -3\cos^{-4}(5x^2+2)\left[-\sin(5x^2+2)\right] \times 10x$$

$$= +30x \frac{\sin(5x^2+2)}{\cos^4(5x^2+2)}$$

11. $y = \tan x \cdot \sin x$

$$\frac{dy}{dx} = \tan x \cdot \cos x + \sin x \cdot \sec^2 x$$

$$= \frac{\sin x}{\cos x} \times \cos x + \sin x \times \frac{1}{\cos^2 x}$$

$$= \frac{\sin x \cdot \cos x}{\cos x} + \frac{\sin x}{\cos^2 x}$$

$$= \sin x + \frac{\sin x}{\cos x} \times \frac{1}{\cos x} \Rightarrow \sin x + \tan x \cdot \sec x$$

12. $y = \tan x(\sec x)$

$$\frac{dy}{dx} = \sec^2(\sec x) \cdot \sec x \cdot \tan x$$

13. $y = \cot^3\left(\frac{x+1}{x-1}\right)$

$$\frac{dy}{dx} = 3\cot^2\left(\frac{x+1}{x-1}\right)\left\{-\text{cosec}^2\left(\frac{x+1}{x-1}\right)\right\}\frac{(x-1)(1)-(x+1)}{(x-1)^2}$$

$$= \frac{6}{(x-1)^2} \cdot \cot^2\left(\frac{x+1}{x-1}\right) \cdot \text{cosec}^2\left(\frac{x+1}{x-1}\right)$$

14. $y = \dfrac{\cos x}{x}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{-x\sin x - \cos x}{x^2}$

15. $y = \tan^{1/2}\sqrt{2x+7}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\cdot\tan^{-\frac{1}{2}}\sqrt{2x+7}\cdot\sec^2\sqrt{2x+7}\times\frac{1}{2}(2x+7)^{-\frac{1}{2}}\cdot(2)$$

$$= \frac{1}{2}\cdot\tan^{-\frac{1}{2}}\sqrt{2x+7}\cdot\sec^2\sqrt{2x+7}\times\frac{2}{2\sqrt{2x+7}}$$

$$= \frac{\sec^2\sqrt{2x+7}}{2\left(\tan^{\frac{1}{2}}\sqrt{2x+7}\right)(2\sqrt{2x+7})}$$

16. $y = x^2\sin x$

$\therefore \quad \dfrac{dy}{dx} = x^2\cos x + \sin x\cdot 2\text{x}$

17. $y = \csc^{-\frac{2}{3}}\sqrt{5x}$

$$y = \csc^{-\frac{2}{3}}(5x)^{\frac{1}{2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{-2}{3}\csc^{-\frac{2}{3}}(5x)^{\frac{1}{2}}\left(-\cot(5x)^{\frac{1}{2}}\right)\cdot\left(\frac{1}{2}(5x)^{-\frac{1}{2}}\right)\cdot(5)$$

$$y' = \frac{5}{3\sqrt{5x}}\cdot\frac{\cot\sqrt{5x}}{\text{cosec}^{\frac{3}{2}}\sqrt{5x}}$$

18. $y = x[\sin(\ln x) + \cos(\ln x)]$

$$y' = x\left[\cos(\ln x)\cdot\frac{1}{x} + -\sin x(\ln x)\frac{1}{x}\right] + [\sin(\ln x) + \cos(\ln x)]\times 1$$

$$= x\frac{\cos(\ln x) - \sin(\ln x)}{x} + \sin(\ln x) + \cos(\ln x)$$

$$= 2\cos(\ln x)$$

19. $y = \sin^{-1} 5x^2$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-(5x^2)^2}}\cdot(10x) = \frac{10x}{\sqrt{1-25x^4}}$$

20. $y = \cot^{-1}\left(\frac{1+x}{1-x}\right)$

$$\left[\because \frac{d}{dx}\cot^{-1} u = -\frac{1}{1+u^2}\frac{du}{dx}\right]$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{1+\left(\frac{1+x}{1-x}\right)^2} \times \frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}$$

$$= -\frac{1}{\frac{(1-x)^2+(1+x)^2}{(1-x)^2}} \times \frac{2}{(1-x)^2}$$

$$= -\frac{(1-x)^2}{(1-x)^2+(1+x)^2} \times \frac{2}{(1-x)^2}$$

$$= \frac{-2}{(1-x)^2+(1+x)^2} \Rightarrow \frac{-2}{1-2x+x^2+1+2x+x^2}$$

$$= \frac{-2}{2+2x^2} = \frac{-2}{2\left(1+x^2\right)} = -\frac{1}{1+x^2}$$

21. $y = \tan^{-1}\sqrt{4x^3-2}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+(\sqrt{4x^3-2})^2} \cdot \frac{1}{2}\left(4x^3-2\right)^{-\frac{1}{2}} \cdot 12x^2$$

$$\left[\because \frac{d}{dx}\tan^{-1} u = -\frac{1}{1+u^2}\frac{du}{dx}\right]$$

$$= \frac{1}{1+\left(4x^3-2\right)} \cdot \frac{6x^2}{\left(\sqrt{4x^3-2}\right)}$$

$$= \frac{6x^2}{\left(\sqrt{4x^3-2}\right)(1+4x^3-2)} = \frac{6x^2}{\left(\sqrt{4x^3-2}\right)\left(4x^3-1\right)}$$

22. $y = \sec^{-1}(3x^2+1)^3$

$$\therefore \quad y' = \frac{1}{\left|(3x^2+1)^3\right|\sqrt{(3x^2+1)^6-1}} \cdot 3(3x^2+1)^2(6x)$$

$$\left[\because \frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}\right]$$

$$= \frac{18(3x^2+1)^2 \cdot x}{((3x^2+1)^3)\sqrt{(3x^2+1)^6-1}} = \frac{18x}{(3x^2+1)\sqrt{(3x^2+1)^6-1}}$$

23. $y = \sin^{-1}\frac{x^2}{2-x} + x^2\sec^{-1}\frac{x}{2}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x^2}{2-x}\right)^2}} \cdot \frac{(2-x)(2x)-x^2(-1)}{(2-x)^2} + x^2 \cdot \frac{1}{\left|\frac{x}{2}\right|\sqrt{\left(\frac{x}{2}\right)^2-1}} \cdot \frac{1}{2}$$

$$+\sec^{-1}\frac{x}{2}\cdot 2x$$

$$\left[\because \frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx} = \frac{d}{dx}\sec^{-1}\cdot\frac{1}{|u|\sqrt{u^2-1}}\right]$$

$$= \frac{1}{\sqrt{1-\left(\frac{x^2}{2-x}\right)^2}} \cdot \frac{4x-2x^2+x^2}{(2-x)^2} + \frac{x^2}{\left|\frac{x}{2}\right|\sqrt{\left(\frac{x}{2}\right)^2-1}} \cdot \frac{1}{2} + \sec^{-1}\frac{x}{2}\cdot 2x$$

$$= \frac{4x-x^2}{\sqrt{\frac{(2-x)^2-(x^2)^2}{(2-x)^2}}} \cdot \frac{1}{(2-x)^2} + \frac{\frac{x^2}{2}}{\left|\frac{x}{2}\right|\sqrt{\left(\frac{x}{2}\right)^2-1}} + 2x\sec^{-1}\frac{x}{2}$$

$$= \frac{4x-x^2}{(2-x)^2\sqrt{\frac{(2-x)^2-x^4}{(2-x)}}} + \frac{\frac{x}{2}\cdot\frac{x^2}{2}}{\sqrt{\frac{x^2}{4}-1}} + 2x\sec^{-1}\frac{x}{2}$$

$$= \frac{4x - x^2}{(2-x)\sqrt{(2-x)^2 - x^4}} + \frac{x}{\sqrt{\frac{x^2-4}{2}}} + 2x\sec^{-1}\frac{x}{2}$$

$$= \frac{4x - x^2}{(2-x)\sqrt{(2-x)^2 - x^4}} + \frac{2x}{\sqrt{x^2-4}} + 2x\sec^{-1}\frac{x}{2}$$

24. $y = \sin^{-1} 2x.\cos^{-1} 2x$

$$\therefore \quad \frac{dy}{dx} = -\sin^{-1} 2x \cdot \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 + \cos^{-1} 2x \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$$

$$= \frac{-2\sin^{-1} 2x}{\sqrt{1-(2x)^2}} + \frac{2\cos^{-1} 2x}{\sqrt{1-(2x)^2}} = \frac{2(\cos^{-1} 2x - \sin^{-1} 2x}{\sqrt{1-4x^2}}$$

25. $y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

$$\ln y = \ln \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$$

$$\ln y = \ln\left(\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}\right)^{\frac{1}{3}}$$

$$\therefore \quad \ln y = \frac{1}{3}[\ln x + \ln(x+1) + \ln(x-2) - \ln(x^2+1) - \ln(2x+3)]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{3}\left[\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right]$$

$$= \frac{1}{3}\sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \cdot \left[\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right]$$

26. $y^{\frac{4}{5}} = \sqrt{\dfrac{\sin x \cdot \cos x}{1+2\ln x}}$

$$\ln y^{\frac{4}{5}} = \ln\sqrt{\frac{\sin x \cdot \cos x}{1+2\ln x}}$$

$$\frac{4}{5}\ln y = \ln\left(\frac{\sin x \cdot \cos x}{1+2\ln x}\right)^{\frac{1}{2}}$$

$$\frac{4}{5}\ln y = \frac{1}{2}[\ln(\sin x) + \ln(\cos x) - \ln(1+2\ln x)]$$

$$\therefore \qquad \frac{dy}{dx} = \frac{5}{8}y\left[\frac{1}{2}\cot x - \frac{1}{2}\tan x - \frac{2}{x(1+2\ln x)}\right]$$

27. $\sqrt{y} = \dfrac{x^5 \tan^{-1} x}{(3-2x)\sqrt[3]{x}}$

$$\ln(y)^{\frac{1}{2}} = \ln\left(\frac{x^5 \tan^{-1} x}{(3-2x)\sqrt[3]{x}}\right)$$

$$\Rightarrow \qquad \frac{1}{2}\ln y = 5\ln x + \ln(\tan^{-1} x) - \ln(3-2x) - \frac{1}{3}\ln x$$

$$\frac{1}{2}\cdot\frac{1}{y}\frac{dy}{dx} = \frac{5}{x} + \frac{1}{\tan^{-1} x}\cdot\frac{1}{1+x^2} - \frac{1}{3-2x}(-2) - \frac{1}{3}\frac{1}{x}$$

$$\frac{dy}{dx} = 2y\left[\frac{5}{x} + \frac{1}{(1+x^2)\tan^{-1} x} + \frac{2}{3-2x} - \frac{1}{3x}\right]$$

$$= 2y\left[\frac{1}{(1+x^2)\tan^{-1} x} + \frac{2}{3-2x} + \frac{15-1}{3x}\right]$$

$$= 2y\left[\frac{1}{(1+x^2)\tan^{-1} x} + \frac{2}{3-2x} + \frac{14}{3x}\right].$$

28. $y = \sec^{-1}(e^{2x})$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{|e^{2x}|\sqrt{(e^{2x})^2 - 1}}\cdot e^{2x}\cdot 2 = \frac{2}{(e^{2x})^2 - 1}$$

29. $y = (\cos x)^{\sqrt{x}}$

$$\ln y = \sqrt{x}\ln(\cos x)$$

$$\frac{1}{y}\frac{dy}{dx} = \sqrt{x}\left(\frac{-\sin x}{\cos x}\right) + \ln(\cos x)\cdot\frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{y}{2\sqrt{x}}[\ln(\cos x) - 2x\tan x]$$

30. $y = (\sin x)\tan x$

$$\ln y = \tan x \ln(\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = \tan x\cdot\frac{\cos x}{\sin x} + \ln(\sin x)\cdot\sec^2 x$$

$$\therefore \quad \frac{dy}{dx} = y\{\tan x\cdot\cot x + \ln(\sin x)\cdot\sec^2 x\}$$

$$= y\{\tan x\cdot\frac{1}{\tan x} + \sec^2 x\ln(\sin x)\}$$

$$= y\{1 + \sec^2 x\cdot\ln(\sin x)\}$$

31. $y = \sqrt{2x^2 + \cosh^2(5x)} = y\left[2x^2 + \cosh^2(5x)\right]^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}[2x^2 + \cosh^2(5x)] - \frac{1}{2}[4x + 2\cosh(5x)\cdot\sinh(5x)\cdot 5]$$

$$= \frac{4x + 10\cosh(5x)\cdot\sinh(5x)}{2\sqrt{2x^2 + \cosh^2(5x)}}$$

$$= \frac{(2x + 5\cosh(5x)\cdot\sinh(5x)}{\sqrt{2x^2 + \cosh^2(5x)}}$$

32. $y = \sinh(\cos 2x)$

$$\frac{dy}{dx} = \cosh(\cos 2x)\cdot(-\sin 2x)\cdot 2 = -2\sin 2x\cosh(\cos 2x)$$

33. $y = \operatorname{csch}\frac{1}{x}$

$$\frac{dy}{dx} = -\operatorname{csch}\frac{1}{x}\cdot\coth\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right) = \frac{1}{x^2}\operatorname{csch}\frac{1}{x}\coth\frac{1}{x}$$

34. $y = x^2 \tan h^2 \sqrt{x}$

$$\therefore \quad \frac{dy}{dx} = x^2 \cdot 2\tanh\sqrt{x} \cdot \sec h^2 \sqrt{x} \cdot \frac{1}{2\sqrt{x}} + 2x\tanh^2\sqrt{x}$$

$$= x\tanh\sqrt{x}\left(\sec h^2\sqrt{x}\,\frac{x}{\sqrt{x}} + 2\tanh\sqrt{x}\right)$$

$$= x\tanh\sqrt{x}\left(\sec h^2\sqrt{x}\cdot\sqrt{x} + 2\tanh\sqrt{x}\right)$$

35. $y = \ln\dfrac{\sin x\cdot\cos x + \tan^3 x}{\sqrt{x}}$

$$y = \ln(\sin x\cdot\cos x + \tan^3 x) - \ln\sqrt{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sin x(-\sin x) + \cos x(\cos x) + 3\tan^2 x\cdot\sec^2 x}{\sin x\cdot\cos x + \tan^3 x} - \frac{1}{2x}$$

$$= \frac{\cos^2 x - \sin^2 x + 3\tan^2 x\sec^2 x}{\sin x\cdot\cos x + \tan^3 x} - \frac{1}{2x}$$

36. $y = \log_4 \sin x$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sin x}\cdot\frac{1}{\ln 4}\cos x = \frac{\cot x}{\ln 4}$$

37. $y = e^{x^2-5x}\cdot(2x - 5e^{5u})$

$$\therefore \quad \frac{dy}{dx} = e^{x^2-5x}\cdot(2x - 5e^{5u})$$

38. $y = e^{x^2}\tan^x$

$$\therefore \quad \frac{dy}{dx} = e^{x^2\tan x}(x^2\sec^2 x + 2x\tan x)$$

39. $y = 7^{\csc\sqrt{2x+3}}$

$$\therefore \quad \frac{dy}{dx} = 7^{\csc\sqrt{2x+3}}(-\csc\sqrt{2x+3}\cdot\cot\sqrt{2x+3})\cdot\frac{2}{2\sqrt{2x+3}}$$

$$= -7^{\csc\sqrt{2x+3}}\,\frac{\csc\sqrt{2x+3}}{\sqrt{2x+3}}\cdot\frac{\cot\sqrt{2x+3}}{1}$$

40. $y = [\ln(x^2+2)^2]\cos x$

$$y = 2\ln(x^2+2)\cdot\cos x$$

$$\therefore \quad \frac{dy}{dx} = 2\ln(x^2+2)\cdot(-\sin x) + \cos x \cdot 2 \cdot \frac{2x}{x^2+2}$$

$$= \frac{4x.\cos x}{x^2+2} - 2\ln(x^2+2)\cdot \sin x$$

41. $y = \sinh^{-1}(\tan x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1+\tan^2 x}}\cdot \sec^2 x \qquad [\text{As } 1+\tan^2 x = \sec^2 x]$$

$$= \frac{\sec^2 x}{\sec x} = \sec x.$$

42. $y = \sqrt{1+(\ln x)^2}$

$$\Rightarrow \quad y = (1+\ln^2 x)^{\frac{1}{2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\cdot(1+\ln^2 x)^{\frac{-1}{2}} \cdot 2\ln x \cdot \frac{1}{x}$$

$$= \frac{2\ln x}{2x(1+\ln^2 x)^{\frac{1}{2}}} = \frac{\ln x}{x\sqrt{1+(\ln x)^2}}$$

43. $y = \dfrac{e^x}{\ln x}$

$$\frac{dy}{dx} = \frac{\ln x \cdot e^x - e^x \cdot \frac{1}{x}}{\ln^2 x}$$

$$= \frac{e^x\left(\ln x - \frac{1}{x}\right)}{\ln^2 x}\cdot\frac{x}{x}$$

$$= \frac{e^x(x\cdot \ln x - 1)}{x\cdot \ln^2 x}$$

44. $y = x^3 \log_2(3-2x)$

$$\therefore \quad \frac{dy}{dx} = x^3 \cdot \frac{1}{3-2x} \cdot \frac{1}{\ln 2} \cdot (-2) + \log_2(3-2x) \cdot 3x^2$$

$$= \frac{-2x^3}{\ln 2(3-2x)} + 3x^2 \cdot \log_2(3-2x)$$

45. $y = 2\cosh^{-1} \cdot \frac{x}{2} + \frac{x}{2}\sqrt{x^2-4}$

$$\therefore \quad \frac{dy}{dx} = 2\frac{1}{\sqrt{\left(\frac{x}{2}\right)^2 - 1}} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{2x}{2\sqrt{x^2-4}} + \frac{1}{2}\sqrt{x^2-4}$$

$$= \frac{1}{\sqrt{\left(\frac{x}{2}\right)^2 - 1}} + \frac{x}{2} \cdot \frac{2x}{2\sqrt{x^2-4}} + \frac{\sqrt{x^2-4}}{2}$$

$$= \frac{2}{\sqrt{x^2-4}} + \frac{x^2}{2\sqrt{x^2-4}} + \frac{\sqrt{x^2-4}}{2}$$

PROBLEM 3.2

Verify the following derivatives:

(*a*) $\frac{d}{dx}\left[5x + \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2\right] = 6 - \frac{1}{x^2}$

(*b*) $\frac{d}{dx}\left[\sqrt{x}(ax^2 + bx + c)\right] = \frac{1}{2\sqrt{x}}(5ax^2 + 3bx + c)$

Solution:

(*a*) $\frac{d}{dx}\left[5x + \left(\sqrt{x + \frac{1}{\sqrt{x}}}\right)^2\right] = 6 - \frac{1}{x^2}$

$$\text{L.H.S.} = 5x + (\sqrt{x})^2 + 2\sqrt{x} \cdot \frac{1}{\sqrt{x}} + \left(\frac{1}{\sqrt{x}}\right)^2$$

$$= 5x + x + 2 + \frac{1}{x}$$

$$= 5x + x + 2 + x^{-1}$$

$$\therefore \quad \frac{d}{dx}(5x + x + 2 + x^{-1}) = 5 + 1 - 1x^{-2}$$

$$= 6 - \frac{1}{x^2} = \text{R.H.S.}$$

(*b*) $\frac{d}{dx}\left[\sqrt{x}(ax^2 + bx + c\right] = \frac{1}{2\sqrt{x}}(5ax^2 + 3bx + c)$

L.H.S. $= \sqrt{x}(ax^2 + bx + c)$

$$= \sqrt{x}ax^2 + \sqrt{x}bx + \sqrt{x}c$$

$$= (x)^{\frac{1}{2}} ax^2 + (x)^{\frac{1}{2}} bx + (x)^{\frac{1}{2}} c$$

$$ax^{\frac{5}{2}} + bx^{\frac{3}{2}} + (x)^{\frac{1}{2}} c$$

$$\therefore \quad \frac{d}{dx}\left[ax^{\frac{5}{2}} + bx^{\frac{3}{2}} + cx^{\frac{1}{2}}\right] = \frac{5}{2}ax^{\frac{3}{2}} + \frac{3}{2}bx^{\frac{1}{2}} + \frac{1}{2}cx^{\frac{-1}{2}}$$

$$= \frac{1}{2}\left(5ax^{\frac{3}{2}} + 3bx^{\frac{1}{2}} + cx^{\frac{-1}{2}}\right) \cdot \frac{\sqrt{x}}{\sqrt{x}}$$

$$\frac{1}{2\sqrt{x}}\left(5ax^{\frac{3}{2}} \cdot x^{\frac{1}{2}} + 3bx^{\frac{1}{2}} \cdot x^{\frac{1}{2}} + cx^{\frac{-1}{2}} \cdot x^{\frac{1}{2}}\right)$$

$$\frac{1}{2\sqrt{x}}(5ax^2 + 3bx + c) = \text{R.H.S.}$$

PROBLEM 3.3

Find the derivative of y with respect to x in the following functions:

(*a*) $y = \frac{u^2}{u^2 + 1}$ and $u = 3x^3 - 2$

(*b*) $y = \sqrt{u} + 2u$ and $u = x^2 - 3$

Solution:

(***a***) $y = \dfrac{u^2}{u^2+1}$ and $u = 3x^3 - 2$

$$\frac{dy}{du} = \frac{(u^2+1)\cdot 2u - u^2 \cdot 2u}{(u^2+1)^2}$$

$$= \frac{2u^3 + 2u - 2u^3}{(u^2+1)^2} = \frac{2u}{(u^2+1)^2}$$

$$= \frac{2y^2}{u^3} = \frac{2y^2}{(3x^2-2)^3}$$

and $\quad u = 3x^3 - 2$

$\Rightarrow \quad \dfrac{du}{dx} = 9x^2$

Now,

$$u = 3x^2 - 2$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{2y^2}{(3x^2-2)^3} = 9x^2 = \dfrac{18x^2 \cdot y^2}{(3x^2-2)^3}$

[**Ans.** $\dfrac{18x^2y^2}{(3x^2-2)^3}$]

(***b***) $y = \sqrt{u} + 2u$ and $u = x^2 - 3$

$$y = (u)^{\frac{1}{2}} + 2u$$

$$\frac{dy}{du} = \frac{1}{2}\cdot u^{-\frac{1}{2}} + 2$$

$$= \frac{1}{2\sqrt{u}} + 2 \Rightarrow \frac{1}{2\sqrt{x^3-3}} + 2$$

$$\frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \left(\frac{1}{2\sqrt{x^3-3}}\right)\cdot 2x$$

$$= \frac{2x}{2\sqrt{x^3-3}} - 4x$$

$$= \frac{x}{\sqrt{x^3-3}} + 4x$$

[**Ans.** $\dfrac{x}{\sqrt{x^2-3}} + 4x$]

PROBLEM 3.4

Find the second derivative for the following functions:

(a) $\boldsymbol{y = \left(x + \frac{1}{x}\right)^3}$

(b) $\boldsymbol{f(x) = \sqrt{2x} + \frac{2\sqrt{2}}{\sqrt{x}}}$ **at** $\boldsymbol{x = 2}$

(c) $\boldsymbol{x^2 - 2xy + y^2 - 16x = 0}$

Solution:

(a) $y = \left(x + \frac{1}{x}\right)^3$

$$\Rightarrow \quad y = \left(x + \frac{1}{x}\right) \cdot (x + \frac{1}{x})2$$

$$\Rightarrow \quad \left(x + \frac{1}{x}\right)\left(x^2 + 2x \cdot \frac{1}{x} + \left(\frac{1}{x}\right)2\right)$$

$$= \left(x + \frac{1}{x}\right)\left(x^2 + 2 + \frac{1}{x^2}\right)$$

$$= x^3 + 2x + \frac{1}{x} + x + \frac{2}{x} + \frac{1}{x^3}$$

$$= x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$$

$$y' = 3x^2 + 3 - 3x^{-2} - 3x^{-4}$$

$$y'' = 6x + \frac{6}{x^3} + \frac{12}{x^5}$$

[**Ans.** $6x + \frac{6}{x^3} + \frac{12}{x^5}$]

(b) $f(x) = \sqrt{2x} + \frac{2\sqrt{x}}{\sqrt{x}}$ at $x = 2$

$$= \sqrt{2}(x)^{\frac{1}{2}} + 2\sqrt{2}(x)^{\frac{-1}{2}}$$

$$y' = \sqrt{2}\,\frac{1}{2}x^{\frac{-1}{2}} + 2\sqrt{2}\left(-\frac{1}{2}\right)x^{\frac{-3}{2}}$$

$$y'' = \left(\frac{1}{2}\right) \cdot \left(\frac{-1}{2}\right)\left(\sqrt{2}\right)x^{\frac{-3}{2}} + 2\sqrt{2}\left(\frac{-1}{2}\right) \cdot \left(\frac{-3}{2}\right)x^{\frac{-5}{2}}$$

$$\Rightarrow \qquad \frac{\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}\cdot\left(\frac{-1}{2}\right)x^{\frac{-3}{2}} - \sqrt{2}\cdot\left(\frac{-3}{2}\right)\cdot x^{\frac{-5}{2}}$$

$$= \frac{1}{\sqrt{2}}\cdot\left(\frac{-1}{2}\right)x^{\frac{-3}{2}} - \sqrt{2}\cdot\left(\frac{-3}{2}\right)\cdot x^{\frac{-5}{2}}$$

at $\qquad x = 2$

$$= \frac{-1}{2\sqrt{2}}\cdot\frac{1}{\sqrt{2^3}} + \frac{3}{\sqrt{2}}\cdot\frac{1}{\sqrt{2^5}} = \frac{1}{4}$$

[**Ans.** $\frac{1}{4}$]

(c) $x^2 - 2xy - y^2 - 16x = 0$

$$(x-y)^2 = 16x$$

$$x - y = \mp 4\sqrt{x}$$

$$\therefore \qquad y = x \mp 4x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 4\left(\frac{1}{2}\right)x^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = \pm 2\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} = \pm\frac{1}{\sqrt{x^3}}$$

[**Ans.** 6]

PROBLEM 3.5

Find the third derivative of the function:
$\boldsymbol{y = \sqrt{x^3}}$

Solution:

$$y = \sqrt{x^3}$$

$$y = x^{\frac{3}{2}}$$

$$y' = \frac{3}{2}x^{\frac{1}{2}}$$

$$y'' = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)x^{-\frac{1}{2}}$$

$$y'' = \frac{3}{4}x^{-\frac{1}{2}}$$

$$y''' = \left(\frac{3}{4}\right)\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$$

$$y''' = \left(-\frac{3}{8}\right)x^{-\frac{3}{2}}$$

$$\therefore \qquad y''' = -\frac{3}{8y}$$

PROBLEM 3.6

Show for $y = \dfrac{u}{v}$ that $y'' = \dfrac{(vu'' - uv'') - 2v'(vu' - uv')}{v^3}$.

Solution:

$$y' = \frac{vu' - uv'}{v^2}$$

$$y'' = \frac{v(vu'' + u'v') - (uv'' + v'u') - 2vv'(vu' - uv')}{v^4}$$

$$y'' = \frac{v(vu'' + u'v' - uv'' + v'u') - 2v'(vu' - uv')}{v^3}$$

$$y'' = \frac{v(vu'' - uv'') - 2v'(vu' - uv')}{v^3}$$

PROBLEM 3.7

Show for $y = u \cdot v$ that $y''' = uv + 3u'v' + 3u''v'' + u'''v$.

Solution:

$$y' = uv' + vu'$$

$$y'' = (uv'' + v'u') + vu'' + u'v'$$

$$y''' = uv'' + v''u' + v'u''' + u'' + u''v' + u'v'' + v'u''$$

$$y''' = uv'' + 3u'v'' + 3u'''v'' + u'''v$$

PROBLEM 3.8

Show that $y = 35x^4 - 30x^2 + 3$ satisfies $(1 - x^2)y'' - 2xy' + 20y + 0$.

Solution:

$$y' = (35)(4)x^3 - 60x$$

$$y' = 140x^3 - 60x$$

$$y'' = (140)(3)x^2 - 60$$

$$y'' = 420x^2 - 60$$

$$\text{L.H.S.} = (1-x^2)y'' - 2xy' + 20y = 0$$

$$\text{L.H.S.} = (1-x^2)(420x^2 - 60) - 2x(140x^3 - 60x) + 20y = 0 = \text{R.H.S.}$$

PROBLEM 3.9

Find $\frac{dy}{dx}$ for the following implicit functions:

(*a*) $x^3 + 4x\sqrt{y} - \frac{5y^2}{x} = 3$

(*b*) $\sqrt{xy} + 1 = y$

(*c*) $3xy = (x^3 + y^3)^{\frac{3}{2}}$

(*d*) $x^3 + x \cdot \tan^{-1} y = y$

(*e*) $\sin^{-1}(xy) = \cos^{-1}(x-y)$

(*f*) $y^2 \cdot \sin(xy) = \tan x$

(*g*) $\sinh y = \tan^2 x$

Solution:

(*a*) $x^3 + 4x\sqrt{y} - \frac{5y^2}{x} = 3$

On differentiating with respect to x,

$$3x^2 + 4\left[x\left(\frac{1}{2}\right)(y)^{-\frac{1}{2}}\frac{dy}{dx} + (y)^{\frac{1}{2}}\right] - \frac{5\left[x\cdot(2y)\frac{dy}{dx} - y^2\cdot 1\right]}{x^2}$$

$$3x^2 + 2x(y)^{-\frac{1}{2}}\frac{dy}{dx} + 4(y)^{\frac{1}{2}} - \frac{10xy\frac{dy}{dx} - 5y^2}{x^2} = 0$$

$$\Rightarrow \qquad 3x^2 + 2xy^{-\frac{1}{2}}\frac{dy}{dx} + 4y^{\frac{1}{2}} - 10xy\cdot x^{-2}\frac{dy}{dx} + 5y^2x^{-2} = 0$$

$$\therefore \qquad \frac{dy}{dx} = \frac{3x^2 + 5y^2x^{-2} + 4\sqrt{y}}{10x^{-1}y - \frac{2x}{\sqrt{y}}}$$

$(b)\ \sqrt{xy}+1=y$

$$(xy)^{\frac{1}{2}}+1=y$$

$$\Rightarrow \quad \frac{1}{2}(xy)^{-\frac{1}{2}}\left(x\frac{dy}{dx}+y\right)=\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{2}(xy)^{-\frac{1}{2}}x\frac{dy}{dx}+\frac{1}{2}(xy)^{-\frac{1}{2}}y=\frac{dy}{dx}$$

$$\therefore \quad \frac{x}{2\sqrt{xy}}\frac{dy}{dx}+\frac{y}{2\sqrt{xy}}\frac{y}{\dfrac{1}{2\sqrt{xy}}}=\frac{dy}{dx}$$

$$\frac{x}{2\sqrt{xy}}=\frac{dy}{dx}-\frac{y}{2\sqrt{xy}}\frac{dy}{dx}$$

$$\frac{y}{2\sqrt{xy}}=\left(1-\frac{x}{2\sqrt{xy}}\right)\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{\dfrac{y}{2\sqrt{xy}}}{\dfrac{2\sqrt{xy}-x}{2\sqrt{xy}}}=\frac{y}{2\sqrt{xy}-x}$$

$(c)\ 3xy=(x^3+y^3)^{3/2}$

On differentiating with respect to x, we obtain

$$3x\frac{dy}{dx}+3y=\frac{3}{2}(x^3+y^3)^{\frac{1}{2}}\left[3x^2+3y^2\frac{dy}{dx}\right]$$

$$3\left[x\cdot\frac{dy}{dx}+y\cdot 1\right]=\frac{3}{2}(x^3+y^3)^{\frac{1}{2}}\cdot\frac{d}{dx}(x^3+y^3)$$

$$=\frac{3}{2}(x^3+y^3)^{\frac{1}{2}}\cdot\left[3x^2+3y^2\frac{dy}{dx}\right]$$

$$3x\frac{dy}{dx}-\frac{3}{2}(x^3+y^3)^{\frac{1}{2}}\cdot 3y^2\frac{dy}{dx}=\frac{3}{2}(x^3+y^3)^{\frac{1}{2}}\cdot 3x^2-3y$$

$$\left[3x-\frac{9}{2}\sqrt{x^3+y^3}\cdot y^2\right]\frac{dy}{dx}=\frac{9}{2}\sqrt{x^3+y^3}\cdot x^2-3y$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{9}{2}x^2\sqrt{x^3+y^3} - 3y}{3x - \frac{9}{2}y^2\sqrt{x^3+y^3}}$$

$$= \frac{3x^2\sqrt{x^3+y^3} - 2y}{2x - 3y^2\sqrt{x^3+y^3}}$$

(*d*) $x^3 + x\tan^{-1} y = y$

$$\frac{dy}{dx} = 3x^2 + x\frac{1}{1+y^2\frac{dy}{dx}} + \tan^{-1} y$$

$$\frac{dy}{dx} = \frac{(1+y^2)(3x^2) + x\frac{dy}{dx} + \tan^{-1} y(1+y^2)}{1+y^2}$$

$$\therefore \qquad (1+y^2)\frac{dy}{dx} - x\frac{dy}{dx} = (1+y^2)(3x^2 + \tan^{-1} y)$$

$$\therefore \qquad \frac{dy}{dx} = \frac{(1+y^2)(3x^2 + \tan^{-1} y)}{1+y^2-x}$$

(*e*) $\sin^{-1}(xy) = \cos^{-1}(x-y)$

$$\frac{1}{\sqrt{1-(xy)^2}}\left(x\frac{dy}{dx} + y\right) = -\frac{1}{\sqrt{1-(x-y)^2}}\left(1-\frac{dy}{dx}\right)$$

$$\frac{x}{\sqrt{1-(xy)^2}}\frac{dy}{dx} + \frac{y}{\sqrt{1-(xy)^2}} = -\frac{1}{\sqrt{1-(xy)^2}} + \frac{1}{\sqrt{1-(x-y)^2}}\frac{dy}{dx}$$

$$\left(\frac{-x}{\sqrt{1-(x-y)^2}} + \frac{1}{\sqrt{1-(x-y)^2}}\right)\frac{dy}{dx}$$

$$= \left(\frac{y}{\sqrt{1-(xy)^2}} + \frac{1}{\sqrt{1-(x-y)^2}}\right)$$

$$\therefore \qquad \frac{dy}{dx} = \frac{y\sqrt{1-(x-y)^2} + \sqrt{1-(xy)^2}}{\sqrt{1-(xy)^2} - x\sqrt{1-(x-y)^2}}$$

(*f*) $y^2 \cdot \sin(xy) = \tan x$

On differentiating, we obtain

$$y^2 \cos(xy)\left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} \cdot \sin(xy) = \sec^2 x$$

$$y^2 \cos(xy)\, x\frac{dy}{dx} + y\cos(xy)y + 2y\frac{dy}{dx}\sin(xy) = \sec^2 x$$

$$\therefore \qquad \frac{dy}{dx} = \frac{\sec^2 x - y^3 \cos(xy)}{2y\sin(xy) + xy^2 \cos(xy)}$$

(g) $\sinh y = \tan^2 x$

$$\cosh y\frac{dy}{dx} = 2\tan x \cdot \sec^2 x$$

$$\therefore \qquad \frac{dy}{dx} = \frac{2\tan x \cdot \sec^2 x}{\cosh y}$$

PROBLEM 3.10

Prove the following formulas:

(*a*) $\frac{d}{dx}\cot u = -\mathrm{cosec}^2 u \cdot \frac{du}{dx}$

(*b*) $\frac{d}{dx}\mathrm{cosec}\, u = -\mathrm{cosecu} \cdot \cot u \cdot \frac{du}{dx}$

(*c*) $\frac{d}{dx}\cos^{-1} u = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

(*d*) $\frac{d}{dx}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

(*e*) $\frac{d}{dx}\sinh u = \cosh u \cdot \frac{du}{dx}$

(*f*) $\frac{d}{dx}\mathrm{cosec}\, u = -\mathrm{cosech}\, u \cdot \coth u \cdot \frac{du}{dx}$

(*g*) $\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{1+u^2}} \cdot \frac{du}{dx}$

(h) $\frac{d}{dx}\operatorname{sech}^{-1}u = -\frac{1}{|u|\sqrt{1-u^2}} \cdot \frac{du}{dx}$

Solution:

(a) $\frac{d}{dx}\cot u = \operatorname{cosec}^2 u \frac{du}{dx}$

$$\text{L.H.S.} = \frac{d}{dx}\left(\frac{\cos u}{\sin u}\right) = \frac{\sin u\left(-\sin u \frac{du}{dx}\right) - \cos u\left(\cos u \frac{du}{dx}\right)}{\sin^2 u}$$

$$= \frac{-\sin^2 u + \cos^2 u}{\sin^2 u}\frac{du}{dx} = -\frac{1\frac{du}{dx}}{\sin^2 u} = -\csc^2 u \frac{du}{dx} = \text{ R.H.S.}$$

(b) $\frac{d}{dx}\operatorname{cosec} u = -\csc u \cdot \cot u \frac{du}{dx}$

$$\text{L.H.S.} = \frac{d}{dx}\left(\frac{1}{\sin u}\right) = \frac{-\cos u}{\sin^2 u}\frac{du}{dx} = \frac{-1}{\sin u} = \frac{\cos u}{\sin u}\frac{du}{dx}$$

$$= -\operatorname{cosec} u \cdot \cot u \frac{du}{dx} = \text{R.H.S.}$$

(c) $\frac{d}{dx}\cos^{-1} u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

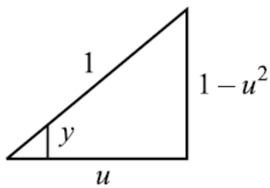
Let,

$$y = \cos^{-1} u \Rightarrow \qquad \therefore u = \cos y$$

$$\frac{du}{dx} = -\sin y \frac{dy}{dx}$$

$$= \frac{-\sqrt{1-u^2}}{1}\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-u}}\frac{du}{dx}$$



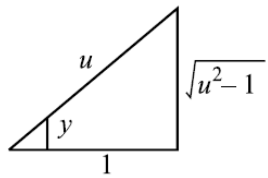
$$\therefore \qquad \frac{d}{dx}(\cos^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

(d) $\dfrac{d}{dx}\sec^{-1}u = \dfrac{1}{|u|\sqrt{u^2-1}}\cdot\dfrac{du}{dx}$

Let $\quad y = \sec^{-1}u \Rightarrow u = \sec y$

$$\frac{du}{dx} = \sec y \cdot \tan y \frac{dy}{dx}$$

$$\frac{du}{dx} = u\sqrt{u^2-1}\,\frac{dy}{dx}$$



$\Rightarrow \qquad \therefore \quad \dfrac{d}{dx}\sec^{-1}u = \dfrac{1}{u\sqrt{u^2-1}}\cdot\dfrac{du}{dx}$

(e) $\dfrac{d}{dx}\sinh u = \cosh u \cdot \dfrac{du}{dx}$

$$\text{L.H.S.} = \frac{d}{dx}\sinh u = \frac{d}{dx}\frac{e^u - e^{-u}}{2} = \frac{1}{2}\left[e^u\frac{du}{dx} - e^{-u}\left(-\frac{du}{dx}\right)\right]$$

$$= \frac{e^u + e^{-u}}{2}\frac{dy}{dx} = \cosh u \frac{dy}{dx} = \text{R.H.S.}$$

(f) $\dfrac{d}{dx}\text{cosech}\, u = \text{cosech}\, u \cdot \coth u \dfrac{du}{dx}$

$$\text{L.H.S.} = \frac{d}{dx}\text{cosech}\, u = \frac{d}{dx}\frac{1}{\sinh u}$$

$$= -\frac{\sin h\, u \cdot 0 - 1 \cdot \cosh x}{\sin h^2 u}$$

$$= -\frac{1}{\sinh u}\cdot\frac{\cosh u}{\sinh u}\frac{du}{dx} = \text{cosech}\, u \cdot \coth u \frac{du}{dx} = \text{R.H.S.}$$

(g) $\dfrac{d}{dx}\sinh^{-1}u = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$

Let $\quad y = \sinh^{-1}u \Rightarrow u = \sinh y$

$$\frac{du}{dx} = \cosh y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cosh y}\cdot\frac{du}{dx}$$

$\therefore$ Since, $\quad \cosh^2 y - u^2 = 1$

$\Rightarrow \quad \cosh^2 y - u^2 = 1$

$$\therefore \quad \Rightarrow \quad \cosh^2 y = 1 + u^2$$

$$\Rightarrow \quad \cosh y = \sqrt{1+u^2}$$

$$\therefore \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}} \cdot \frac{du}{dx}$$

(h) $\dfrac{d}{dx}\operatorname{sech}^{-1} u = -\dfrac{1}{|u|\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$

Let $\quad y = \operatorname{sech}^{-1} u \Rightarrow u = \operatorname{sech} y$

$$\frac{du}{dx} = -\operatorname{sech} \cdot \tan y \frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{\operatorname{sech} y \cdot \tanh y} \frac{du}{dx}$$

Since, $\quad \operatorname{sech}^2 y + \tanh^2 y = 1$

$$u^2 + \tanh^2 y = 1$$

$$\therefore \quad \tanh^2 y = \sqrt{1-u^2}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{|u|\sqrt{1-u^2}} \cdot \frac{du}{dx} = \text{R.H.S.}$$

PROBLEM 3.11

Show that the tangent to the hyperbola $x^2 - y^2 = 1$ at the point $P(\cosh u,\ \sinh u)$, cuts the x-axis at the point $(\operatorname{sech} u,\ 0)$ and except when vertical, cuts the y-axis at the point $(0,\ \operatorname{cosech} u)$.

Solution: $x^2 - y^2 = 1 \Rightarrow 2x - 2yy' = 0$

$$\therefore \quad y' = \frac{2x}{2y}$$

The slope at $p(\cosh u \cdot \sinh u)$ is

$$m = \frac{\cosh u}{\sinh u} = \coth u$$

$$y - \sinh u = \coth u(x - \cosh u)$$

$$y - \sinh u = \frac{\cosh u}{\sinh u}(x - \cosh u)$$

$$y - \sinh u = \frac{x\cosh u - \cosh^2 u}{\sinh u}$$

$$y\sinh u - \sinh^2 u + \cos^2 \mathrm{h}^2 u = x\cosh u$$

$\therefore$ $$y\sinh u = x\cosh u - 1$$

At $$\mathrm{al}\, y = o \Rightarrow x = \operatorname{sech} u$$

At $$\mathrm{al}\, x = o \Rightarrow y = \operatorname{csch} u$$

CHAPTER 4

APPLICATIONS OF DERIVATIVES

PROBLEMS

PROBLEM 4.1

Find the velocity v if a particle's position at time t is $s = 180t - 16t^2$. When does the velocity vanish? Solution:

$$s = 180t - 16t^2$$

$$v = \frac{ds}{dt} = 180 - 32t \Rightarrow 180 - 32t = 0 \Rightarrow 180 = 32t$$

$$\therefore \quad t = \frac{180}{32} = 5.625\,\text{sec}$$

(Ans. 5.625)

PROBLEM 4.2

If a ball is thrown straight up with a velocity of 32 *ft*/sec, its height after *t* sec is given by the equation $s = 32t - 16t^2$. At what instant will the ball be at its highest point? And how high will it rise?

Solution:

$$v = 32\ ft/\text{sec}, s = 32t - 16t^2$$

$$\frac{dv}{dt} = 32 - 32t \Rightarrow 32 - 32t = 0 \therefore t = 1\,\text{sec}$$

$$s = 32t - 16t^2 = 32(1) - 16(1)^2$$

$$s = 32 - 16 = 16\ ft$$

(Ans. 1, 16)

PROBLEM 4.3

A stone is thrown vertically upwards at $35\,\text{m/sec}$. Its height is $s = 35t - 4.9t^2$ in meters above the point of projection, where t is time in seconds later.

(*a*) What is the distance moved, and the average velocity during the 3^{rd} sec? (from $t = 2$ to $t = 3$)?

(*b*) Find the average velocity for the intervals $t = 2$ to $t = 2.5$, $t = 2$ to $t = 2.1$; and $t = 2$ to $t = 2 + h$.

(*c*) Deduce the actual velocity at the end of the 2^{nd} sec.

Solution:

(*a*) $v = 35\,\text{m/sec},\ s = 35t - 4.9t^2$

$$v_{av} = \frac{\Delta s}{\Delta t} \Rightarrow \Delta s = s(3) - s(2)$$

$$-(35(2) - 4.9(2)^2) = (35(3) - 4.9(3)^2)$$

$$\Delta s = (105 - 44.1) - (70 - 19.6)$$

$$= 60.9 - 50.4 = 10.5\,\text{m}$$

$$v_{av} = \frac{10.5}{3-2}$$

$$= 10.5\,\text{m/sec}$$

(*b*) $$\frac{\Delta s}{\Delta t} = \frac{(35(2.5) - 4.9(2.5)^2 - (35(2) - 4.9(2)^2)}{2.5-2} = 12.95\,\text{m/sec}$$

$$v_{av} = \frac{\Delta s}{\Delta t} = \frac{(35(2.1) - 4.9(2.1)^2 - (35(2) - 4.9(2)^2)}{2.1-2} = 14.91\,\text{m/sec}$$

$$v_{av} = \frac{\Delta s}{\Delta t} = \frac{(35(2+h) - 4.9(2+h)^2 - (35(2) - 4.9(2)^2)}{2.5-2}$$

$$v_{av} = \frac{(70 + 35h - 4.9(h^2 + 4h + 4) - (70 - 19.6))}{(2+h)-2}$$

$$= \frac{(70 + 35h - 4.9h^2 - 19.6h - 19.6) - 70 + 19.6)}{h} = \frac{15.4h - 4.9h^2}{h}$$

$$= \frac{h(15.4 - 4.9h)}{h} = 15.4 - 4.9h\,\text{m/sec}$$

(c) At the end, the height $h = 0$

$15.4 - 4.9h = 15.4 - 4.9(0) = 15.4\,\text{m/sec}$

(Ans. (a) 10.5, 10.5; (b) 12.95, 14.91, 15.4 – 4.9, (c) 15.4)

PROBLEM 4.4

A stone is thrown vertically upwards at $24.5\,\text{m/sec}$ from a point just a little higher than cliff's ledge. Its height above the ledge t sec later is $4.9t(5-t)\,\text{m}$. If its velocity is $v\,\text{m/sec}$, differentiate to find v in terms of t:

(i) When is the stone at the ledge level?

(ii) Find its height and velocity after 1, 2, 3, and 6 seconds.

(iii) What meaning is attached to a negative value of s? A negative value of v?

(iv) When is the stone momentarily at rest? What is the greatest height reached?

(v) Find the total distance moved during the 3rd second.

Solution:

$$v = 24.5\,\text{m / s}$$

$$s = 4.9\,t(s-t)$$

$\therefore$ Average rate of change

$$v = \lim_{\Delta s\to 0}\frac{\Delta s}{\Delta t} = \lim_{\Delta s\to 0}\frac{4.9(t+\Delta t)(s-t-\Delta t)-4.9t(s-t)}{\Delta t}$$

$$s = 4.9t(s-t) = 24.5t - 4.9t^2$$

$$\lim_{\Delta s\to 0}\frac{\Delta s}{\Delta t} = \lim_{\Delta t}\frac{s(t+\Delta t)-s(t)}{\Delta t}$$

$$= \lim_{\Delta s\to 0}\frac{24\cdot s(t-\Delta t)-4.9(t+\Delta t)^2-24.5t-4.9t^2}{\Delta t}$$

$$\Rightarrow \quad = \lim_{\Delta s\to 0}\frac{24\cdot st+24\cdot s\Delta t-4.9t^2-9.8t\Delta t+4.9t^2-24\cdot st}{\Delta t}$$

$$= \lim_{\Delta s\to 0}\frac{24\cdot st+24\cdot s\Delta t-4.9t^2-9.8t\Delta t+4.9t^2-24\cdot st-4.9t^2}{\Delta t}$$

$$= \lim_{\Delta s\to 0}\frac{24\cdot s\Delta t-9.8t\Delta t}{\Delta t} = \frac{(24\cdot s-9.8t)\Delta t}{\Delta t}$$

$$\therefore \quad v = 24\cdot s - 9.8t$$

(i) when $v=0$

$$v=24\cdot s-9.8t=0$$

$$24\cdot s-9.8t=0\Rightarrow 24\cdot s=9.8t$$

$$\therefore \qquad t=2.5\,\text{sec}.$$

(ii) $s(1)=4.9(1)(s-1)=19.6\,\text{m}$

and $$v=24\cdot s-9.8(1)=14.7\text{ m/s}$$

$$s(2)=4.9(2)(s-2)=29.9\,\text{m}$$

and $$v=24\cdot s-9.8(2)=4.9\,\text{m/s}$$

$$s(3)=4.9(3)(s-3)=29.9\,\text{m}$$

and $$v=24\cdot s-9.8(3)=4.9\,\text{m/s}$$

$$s(6)=4.9(6)(s-6)=-29.4\,\text{m}$$

and $$v=24\cdot s-9.8(6)=-34.3\,\text{m/s}$$

(iii) The negative valve of S means that the stone is below ledge; the negative valve for v means a storm is blowing.

(iv) $$v=0\Rightarrow 24\cdot s-9.8(t)=0$$

$$\therefore \qquad t=2\cdot s$$

$$s=4.9(2\cdot s)(s-2\cdot s)=30.625\,\text{m}$$

(v) $(30.62s-29.4)=2.4s\,\text{m}$, the total distance traveled during the third second.

(Ans. $v=24.5-9.8t$; (i) 0, 5; (ii) 19.6, 29.4, 29.4, −29.4; (iii) 14.7, 4.9, −4.9, −34.3; (iv) 2.5; 30.625; (v) 2.45)

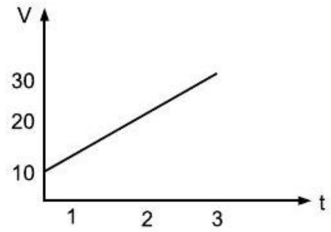
PROBLEM 4.5

A stone is thrown vertically downwards with a velocity of $10\,\text{m/sec}$, and gravity produces on it an acceleration of $9.8\,\text{m/sec}^2$:
(a) What is the velocity after 1, 2, and 3 t sec?
(b) Sketch the velocity-time graph.

Solution:

$$v=10\frac{m}{s};\ a=9.8\,\text{m/s}^2$$

The velocity after time t is $10+9.8t$.



t	1	2	3	6
v	19.8	29.6	39.4	$10+9.8t$

(Ans. $19.8, 29.6, 39.4, 10+9.8t$)

PROBLEM 4.6

A car accelerates from 5 km/h to 41 km/h in 10 sec. Express this acceleration in (*i*) km/h per sec., (*ii*) m/sec.^{2}, and (*iii*) km/h^2.

Solution:

$$(i)\quad a=\frac{\Delta v}{\Delta t}=\frac{41-5}{10}=3.6\frac{\text{km}}{h}\text{per sec}$$

$$(ii)\quad a=\frac{3.6\times 1000}{3600}=1\,\text{m/s}^2 \qquad [\because\ 1h=3600s, 1\,\text{km}=1000\,\text{m}]$$

$$(iii)\quad a=3.6\times 3600=1296\,\text{km}/h^2$$

(Ans. (*i*) 3.6; (*ii*) 1; (*iii*) 12960)

PROBLEM 4.7

A car can accelerate at 4 m/sec^2. How long will it take to reach 90 km/h from rest?

Solution:

$$a=4\,\text{m/sec}^2,\ \text{velocity}=90\,\text{km/h}$$

$$v=\frac{90(1000)}{3600}=25\,\text{m/s}$$

$$a=\frac{\Delta v}{\Delta t}=\frac{25}{\Delta t}=4\Rightarrow \Delta t=6.25\,\text{sec}$$

(Ans. 6.25)

PROBLEM 4.8

An express train reducing its velocity to 40 km/h has to apply the brakes for 50 sec. If the retardation produced is 0.5 m/sec^2, find its initial velocity in km/h.

Solution:

$$\text{Velocity} = 40\,\text{km/h}\; t = 50\,\text{sec}$$

$$a = 0.5\,\text{m/sec}^2$$

$$a = \frac{\Delta v}{\Delta t} = 0.5 = \frac{\Delta v}{50} = \Delta v = 25\,\text{m/s} = \frac{25}{1000} \times 3600$$

$$= 90\,\text{km}/h$$

$$\Delta v = v_1 - v_2 = 90 = v_1 - 40 \Rightarrow 130\,\text{km}/h$$

(Ans. 130)

PROBLEM 4.9

At the instant from which time is measured, a particle is passing through *O* and traveling towards *A*, along the straight line *OA*. It is *s m*, from *O* after *t* sec, where $s = t(t-2)^2$:

(*i*) When is it again at *O* ?
(*ii*) When and where is it momentarily at rest?
(*iii*) What is the particle's greatest displacement from *O*, and how far does it move during the first 2 seconds?
(*iv*) What is the average velocity during the 3$^{\text{rd}}$ second?
(*v*) At the end of the 1$^{\text{st}}$ second, where is the particle, which way is it going, and is its speed increasing or decreasing?
(*vi*) Repeat for the instant when $t = -1$.

Solution:

(*i*) $t = 0\; t = 2\,\text{sec}$

At $t = 2$, for the particle at 0,

(*ii*)
$$v = t\,(t-2)^2 = t\,(t^2 - 4t + 4)$$

$$= t^3 - 4t^2 + 4t$$

$$\therefore \quad \frac{dv}{dt} = 3t^2 - 0t + 4$$

$$(3t-2)(t-2) = 0 \quad \Rightarrow \therefore \text{ either}\,(t-2) = 0$$

$$\therefore \qquad t = 2$$

Or $(3t-2)=\therefore t=\dfrac{2}{3}$

$\therefore$ At $t=2 \quad \Rightarrow \quad s=t(t-2)^2=2(2-2)^2=0$

At $\quad t=\dfrac{2}{3} \quad \Rightarrow \quad s=\dfrac{2}{3}\left(\dfrac{2}{3}-2\right)^2=\dfrac{32}{27}$ m

(iii) The particle's greatest displacement from 0 is $\dfrac{32}{27}m,h,t$ during the first 2 seconds.

$$2\times\frac{32}{27}=\frac{64}{27}m$$

(iv) $V_{av}=\dfrac{\Delta s}{\Delta t}=\dfrac{s(3)-s(2)}{3-2}=\dfrac{3(3-2)^2-2(2-2)^2}{1}=3\,\text{m/s}$

(v) At $t=1, s=1(1-2)^2=1$ m from 0, it is going from O to A (lie OA)

$v=(3t-2)(t-2)=(3(1)-2)(1-2)=-1\,\text{m/s}$

$v'=3t^2-8t+4 \qquad \therefore a=6t-8$ at $t=1$

$=6-8\,\text{m/s}^2\; a, v<0$

(vi) At $t=-1\Rightarrow s=-(-1-2)^2=-90$ m from O (i.e., $9n$ from O on AO); it is going from O to the negative side of A.

$$v=(-1-2)(3(-1)-2)=15\,\text{m/s}$$

$$a=6(-1)-8=-14\,\text{m/s}^2$$

Since $v>0$ and $a<0$, the speed is decreasing.

(Ans. (i) 2; (ii) 0,32/27; (iii) 64/27; (iv) 3; (v) *OA*; inceasing; (vi) *AO*; decreasing)

PROBLEM 4.10

A particle moves in a straight line so that after t sec it is $s\,m$, from a fixed point O on the line, where $s=t^4+3t^2$. Find

(i) the acceleration when $t=1, t=2$, and $t=3$.
(ii) the average acceleration between $t=1$ and $t=3$.

Solution:

$(i)\quad t=1, t=2,$

$$s=t^4+3t^2 \quad\Rightarrow\quad v=4t^3+6t \Rightarrow a=12t^2+6$$

$$a=12t^2+6$$

$$t=1, a=12+6=18\,\text{m/s}^2$$

$$t=2, a=12(2)^2+6=12\times4+6=54\,\text{m/s}^2$$

$$t=3, a=12(3)^2+6=114\,\text{m/s}^2$$

$$(ii)\quad a_{av}=\frac{\Delta v}{\Delta t}=\frac{v(3)-v(1)}{3-1}$$

$$=\frac{4\times3^3+6\times3-(4\times1+6\times1)}{2}=58\,\text{m/s}^2$$

(Ans. (*i*) 18, 54, 114; (*ii*) 58)

PROBLEM 4.11

A particle moves along the x-axis in such a way that its distance x cm from the origin after t sec is given by the formula $x=27t-2t^2$, what are its velocity and acceleration after 6.75 sec? How long does it take for the velocity to be reduced from 15 cm/sec to 9 cm/sec, and how far does the particle travel mean while?

Solution:

$$x=27t-2t^2\ v\ ?\ a\ ?\ t=6.75\,\text{sec}$$

$$v=27-4t \text{ and } a=-4$$

(i) At $t = 6.75$ $\quad v=27-4(6.75)=0\,\text{cm/s};\ a=-4\,\text{cm/s}^2$

(ii) As, $\quad v=27-4t;\ 15=27-4t\ \ t=3\,\text{sec}$

and $\quad 9=27-4t \Rightarrow t=4.5$

$\therefore \quad \Delta t=4.5-3=1.5\,\text{sec}$

(iii) $\quad x=27\times1.5-2\times(1.5)^2=36\,\text{cm}$

(Ans. 0, −4, 1.5; 36)

PROBLEM 4.12

A point moves along a straight line OX so that its distance x cm from the point O at time t sec is given by the formula $x=t^3-6t^2+9t$. Find:

(*i*) at what times and in what positions the point will have zero velocity.
(*ii*) its acceleration at these instants.
(*iii*) its velocity when its acceleration is zero.

Solution:

(*i*) $x = t^3 - 6t^2 + 9t$ find

$v = 3t^2 - 12t + 9 = 0 \Rightarrow (t-1)(t-3) = 0$

either $t = 1 \text{ or } t = 3$

$x(1) = 1 - 6\times1 + 9\times1 = 4\,\text{cm}$ and

$x(3) = 27 - 6\times9 + 9\times3 = 0\,\text{cm}$

(*ii*) $a = 6t - 12$ at $t = 1 \Rightarrow a = 6\times1 - 12 = -6\,\text{cm}/s$

at $t = 3 \quad \Rightarrow \quad a = 6\times3 - 12 = 6\,\text{cm}/s$

(*iii*) $a = 6t - 12 = 0 \Rightarrow t = 2$

$v(2) = 3\times4 - 12\times2 + 9 = -3\,\text{cm}/s$

(Ans. (*i*) 1, 3; 4, 0; (*ii*) –6, 6; (iii) –3)

PROBLEM 4.13

A particle moves in a straight line so that its distance x cm from a fixed point O on the line is given by $x = 9t^2 - 2t^3$, where t is the time in seconds measured from O. Find the speed of the particle when $t = 3$. Also find the distance from O of the particle when $t = 4$, and show that it is then moving towards O.

Solution:

$x = 9t^2 - 2t^3$

$v = 18t - 6t^2 \Rightarrow v(3) = 8\times3 - 6\times9 = 0\,\text{cm}/s$

$x(4) = 9\times16 - 2\times64 = 16\,\text{cm}$

$v(4) = 18\times4 - 6\times16 = -24\,\text{cm}/s$

$v(4) < 0$, hence the particle is moving towards O.

(Ans. 0, 16)

PROBLEM 4.14

Find the limits for the following functions using L'Hopital's rule:

(1) $\lim\limits_{x\to\infty} \dfrac{5x^2 - 3x}{7x^2 + 1}$ **(2)** $\lim\limits_{t\to0} \dfrac{\sin t^2}{t}$

(3) $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{2x-\pi}{\cos x}$ (4) $\lim\limits_{t\to 0} \dfrac{\cos t-1}{t^2}$

(5) $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{1-\sin x}{1+\cos 2x}$ (6) $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\sin x-\cos x}{x-\frac{\pi}{4}}$

(7) $\lim\limits_{x\to 1} \dfrac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$ (8) $\lim\limits_{x\to 0} \dfrac{x(\cos x-1)}{\sin x - x}$

(9) $\lim\limits_{x\to 0} x.\csc^2\sqrt{2x}$ (10) $\lim\limits_{x\to 0} \dfrac{\sin x^2}{x\cdot \sin x}$

$$\left(\textbf{Ans.}(1)\frac{5}{7};(2)0;(3)-2;4(4)-\frac{1}{2};(5)\frac{1}{4};(6)\sqrt{2};(7)-1;(8)3;(9)\frac{1}{2};(10)1\right)$$

Solution:

1. $\lim\limits_{x\to\infty} \dfrac{5x^2-3x}{7x^2+1} \overset{\frac{0}{0}}{\Rightarrow}$ Use L'Hopital's rule

$$=\lim_{x\to\infty}\frac{10x-3}{14x}\overset{\frac{\infty}{\infty}}{\Rightarrow}\lim_{x\to\infty}\frac{10}{14}=\frac{5}{7}$$

2. $\lim\limits_{t\to 0}\dfrac{\sin t^2}{t}\overset{\frac{0}{0}}{\Rightarrow}\lim\limits_{t\to 0}\dfrac{\cos t^2\cdot 2t}{1}=\cos(0)\cdot 2(0)=1\times 0=0$

3. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{2x-\pi}{\cos x}=\dfrac{2\frac{\pi}{2}-\pi}{\cos\frac{\pi}{2}}=\overset{\frac{0}{0}}{\Rightarrow}$ use L'Hopital's rule

$$=\lim_{x\to\frac{\pi}{2}}\frac{2}{-\sin x}=\frac{2}{-\sin\frac{\pi}{2}}=-2$$

4. $\lim\limits_{t\to 0}\dfrac{\cos t-1}{t^2}\overset{\frac{0}{0}}{\Rightarrow}\lim\limits_{t\to 0}\dfrac{-\sin t}{2t}\overset{\frac{0}{0}}{\Rightarrow}\lim\limits_{t\to 0}\dfrac{-\cos t}{2}=-\dfrac{1}{2}$

5. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1-\sin}{1+\cos 2x}\overset{\frac{0}{0}}{\Rightarrow}\lim\limits_{x\to\frac{\pi}{2}}\dfrac{-\cos x}{-2\sin 2x}\overset{\frac{0}{0}}{\Rightarrow}\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\sin x}{-4\cos 2x}=\dfrac{\sin\frac{\pi}{2}}{-4\cos\pi}=\dfrac{1}{4}$

6. $\lim_{x\to\frac{\pi}{2}}\frac{\sin x-\cos x}{x-\frac{\pi}{2}}\overset{\frac{0}{0}}{\Rightarrow}\lim_{x\to\frac{\pi}{2}}\frac{\cos x+\sin x}{1}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}$

7. $\lim_{x\to1}\frac{2x^2-(3x+1)\sqrt{x}+2}{x-1}=\frac{2(1)^2-(3+1)\sqrt{1}+2}{1-1}$

$=\frac{2-4+2}{1-1}=\frac{0}{0}$ ∴ Use L'Hopital's rule

$$\lim_{x\to1}\frac{4x-(3x+1)\frac{1}{2}x^{-\frac{1}{2}}-\sqrt{x}\cdot 3}{1}$$

$$\lim_{x\to1}\frac{4x-\left[3\sqrt{x}+\frac{3x+1}{2\sqrt{x}}\right]}{1}$$

$$=\frac{4(1)-\left[(3\sqrt{1})+\frac{3(1)+1}{2\sqrt{1}}\right]}{1}$$

$$=\frac{4-\left(3+\frac{4}{2}\right)}{1}$$

$$=\frac{4-5}{1}=-1$$

8. $\lim_{x\to0}\frac{x(\cos x-1)}{\sin x-x}=\frac{0}{\sin 0-0}=\frac{0}{0}$ ∴ use L'Hopital's rule

$\Rightarrow \lim_{x\to0}\frac{x(-\sin x)+(\cos x-1)(1)}{\cos x-1}=\frac{0(-\sin 0)+(\cos 0-1)}{\cos 0-1}=\frac{0}{0}$

$\Rightarrow \lim_{x\to0}\frac{x(-\cos x)+(-\sin x)(1)-\sin x}{-\sin x}\times -1$

$$\lim_{x\to0}\frac{x\cos x+2\sin x}{\sin x}=\frac{0\cos 0+\sin 0}{\sin 0}=\frac{0}{0}$$

$$\lim_{x\to0}\frac{x(-\sin x)+\cos x+2\cos x}{\cos x}=\frac{0(-\sin 0)+\cos 0+2\cos 0}{\cos 0}$$

$$=\frac{0+1+2}{1}=3$$

9. $\lim_{x\to 0} x\csc^2\sqrt{2x} = 0\csc^2\sqrt{2\times 0} = 0\times\infty \therefore$ use L'Hopital's rule

$$\lim_{x\to 0}\frac{x}{\sin^2\sqrt{2x}} = \frac{0}{0} \quad \therefore \quad \lim_{x\to 0}\frac{1}{2\sin\sqrt{2x}\cdot\cos\sqrt{2x}\cdot\dfrac{2}{2\sqrt{2x}}}$$

$$\lim_{x\to 0}\frac{\sqrt{2x}}{\sin 2\sqrt{2x}} = \frac{0}{0} \qquad [\because 2\sin\sqrt{2x}\cdot\cos\sqrt{2x} = \sin 2\sqrt{2x}]$$

$$\lim_{x\to 0}\frac{\dfrac{2}{2\sqrt{2x}}}{\cos 2\sqrt{2x}\times\dfrac{2\times 2}{2\sqrt{2x}}} = \lim_{x\to 0}\frac{1}{2\cos 2\sqrt{2x}} = \frac{1}{2\cos 0} = \frac{1}{2}$$

10. $\lim_{x\to 0}\dfrac{\sin x^2}{x\sin x} = \dfrac{\sin^2(0)}{0\sin 0} = \dfrac{0}{0}$ use L'Hopital's rule

$$\therefore \lim_{x\to 0}\frac{\cos x^2\cdot 2x}{x(\cos x)+\sin x\cdot 1} = \frac{0}{0}$$

$$\therefore \lim_{x\to 0}\frac{\left(-2x\sin x^2\right)+\cos x^2\cdot 2}{(-\sin x)+\cos x+\cos x} = \frac{0+2}{0+1+1} = \frac{2}{2} = 1$$

PROBLEM 4.15

Find any local maximum and local minimum values, then sketch each curve by using the first derivative.

(1) $f(x) = x^3 - 4x^2 + 4x + 5$ **(Ans. max.(0.7, 6.2); min.(2, 5))**

(2) $f(x) = \dfrac{x^2-1}{x^2+1}$ **(Ans. min. (0, –1))**

(3) $f(x) = x^5 - 5x - 6$ **(Ans. max. (–1, –2) ; min. (1, –10))**

(4) $f(x) = x^{\frac{4}{3}} - x^{\frac{1}{3}}$ **(Ans. min. (0.25, –0.47))**

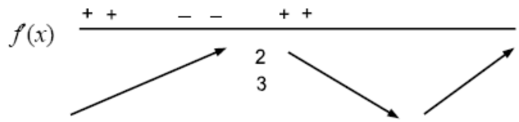
Solution:

(1)

$$f(x) = x^3 - 4x^2 + 4x + 5$$

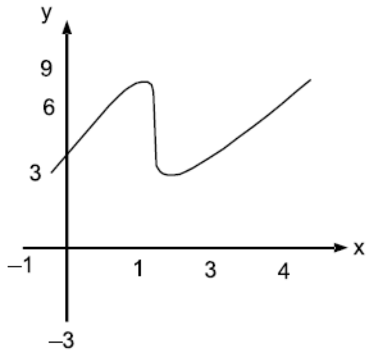
$$f'(x) = 3x^2 - 8x + 4 = 0$$

$$\Rightarrow \quad (3x-2)(x-2) = 0 \Rightarrow \quad \therefore x = \frac{2}{3}, 2$$



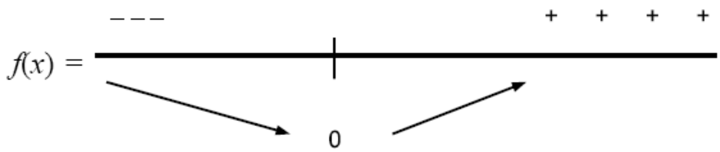
$\therefore$ The function has a local max at $x = \dfrac{2}{3}$ and local min at $x = 2$.

x	$f(x)$
0	5
$\frac{2}{3}$	6.2
1	6
2	5
3	8

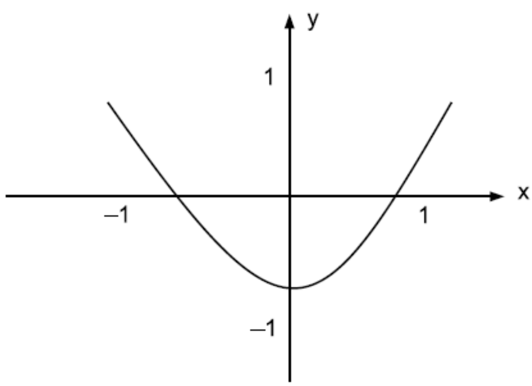


$$(2)\quad f(x) = \frac{x^2 - 1}{x^2 + 1} \Rightarrow f'(x) = \frac{(x^2+1)\cdot(2x) - (x^2-1)(2x)}{(x^2+1)^2}$$

$$= \frac{2x^3 + 2x - 2x^3 + 2x}{\left(x^2+1\right)^2} = \frac{4x}{\left(x^2+1\right)^2} = 0 \Rightarrow x = 0$$



x	$f(x)$
–1	0
0	–1
1	0

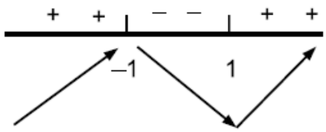


(3) $$f(x) = x^5 - 5x - 6 \Rightarrow f'(x) = 5x^4 - 5 = 5(x^4 - 1) = 0$$

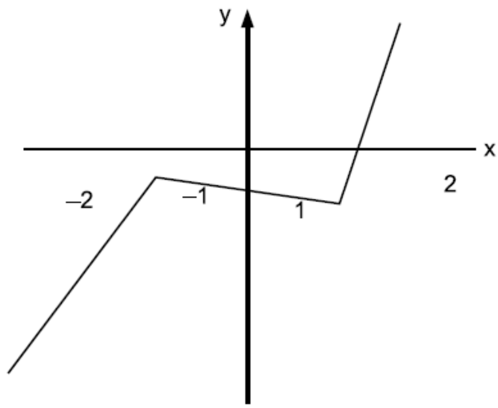
$$\Rightarrow \quad 5(x-1)(x+1)(x^2+1) = 0$$

$$\therefore \quad \Rightarrow \quad x = 1, -1$$

$\therefore$ The function has a local max at $x = -1$ and local min at $x = 1$.

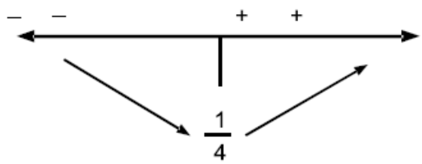


x	$f(x)$
–2	–28
–1	–2
0	–6
1	–10
2	21



(4) $f'(x) = \frac{4}{3}x^{\frac{1}{3}} - \frac{1}{3}x^{-\frac{2}{3}} = 0$

$$\Rightarrow \quad \frac{1}{3x^{\frac{2}{3}}}(4x-1) = 0 \Rightarrow x = \frac{1}{4}$$



The function has local minimum at $x = \frac{1}{4}$.

x	$f(x)$
–1	2
0.25	–0.47
1	0

PROBLEM 4.16

Find the interval of the x-values on which the curve is concave up and concave down, then sketch the curve:

(1) $f(x)=\dfrac{x^3}{3}+x^2-3x$

(2) $f(x)=x^2-5x+6$

(3) $f(x)=x^3-2x^2+1$

(4) $f(x)=x^4-2x^2$

Solution:

(1) $$f(x)=\frac{x^3}{3}+x^2-3x$$

$$f'(x)=\frac{3}{3}x^2+2x-3=0 \Rightarrow (x+3)(x-1)=0$$

$\therefore$ $$x=-3,1$$

$$f''(x)=2x+2=0 \Rightarrow x=-1 \text{ inflection point}$$

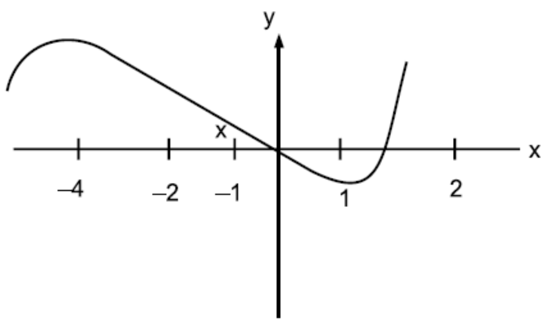
$$f''(1)=2+2=4>0 \text{ concave up}$$

$$f''(2)=-6+2=4<0 \text{ concave down}$$

(Ans. up $(-1,\infty)$; down $(-\infty,-1)$)

$\therefore$ The interval of the x-values in which the curve is concave up at $(-1,\infty)$ and concave down at $(-\infty,-1)$.

x	$f(\infty)$
–4	6.9
–3	9
–1	3.7
1	–1.7
2	0.3

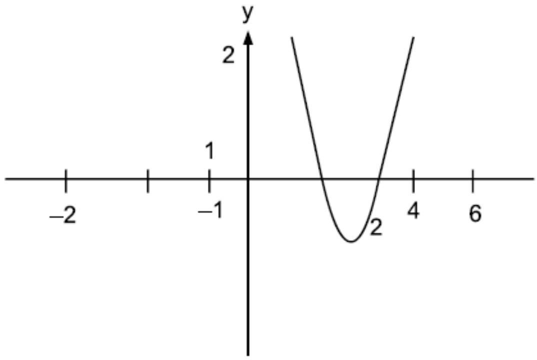


(2) $f(x)=x^2-5x+6 \Rightarrow f'(x)=2x-5=0 \Rightarrow x=2.5$

$f^2(x)=2>0 \quad \Rightarrow \quad$ Concave up the curve

x is concave up when the value is up

x	$f(\infty)$
1	2
2.5	–0.25
–1	3.7
1	–1.7
2	0.3



(Ans. up $(-\infty, \infty)$)

(3) $f(x) = x^3 - 2x^2 + 1 \Rightarrow f'(x) = 3x^2 - 4x = 0$

$\Rightarrow \quad x(3x-4) = 0 \Rightarrow x = 0, \frac{4}{3}$

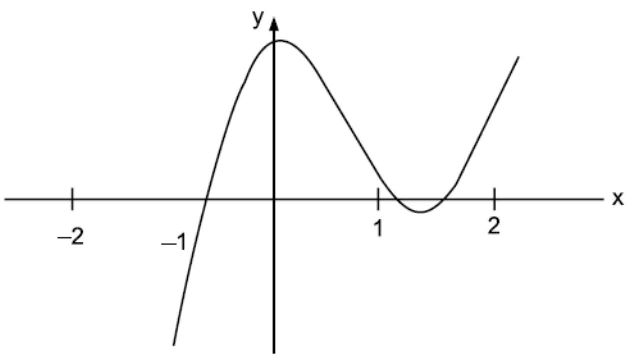
$$f''(x) = 6x - 4 = 0 \Rightarrow x = \frac{2}{3} \text{ inflection point}$$

$$f''(0) = 0 - 4 < 0 \Rightarrow \text{and } f''\left(\frac{4}{3}\right) = 8 - 4 > 0 \text{ concave up}$$

$\therefore$ The interval of the x-values in which the curve is

Concave up at $\left(\frac{2}{3}, \infty\right)$ and concave down at $\left(-\infty, \frac{2}{3}\right)$.

x	$f(x)$
–1	–2
0	1
0.7	0.4
1.3	–0.2
1	2



(4) $f(x) = x^4 - 2x^4 \Rightarrow f'(x) = 4x^3 - 4x = 0$

$\Rightarrow \quad 4x(x^2 - 1) = 0 \quad \Rightarrow \quad x = 0, -1, 1$

$f''(x) = 12x^2 - 4 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}} \text{inf} \, lection \, po \, \text{int} \, s$

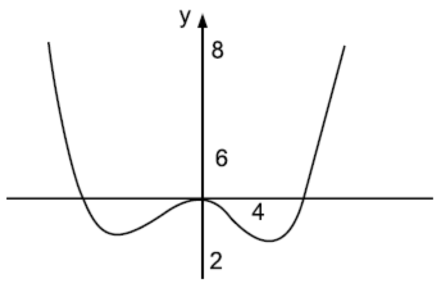
$f''(-1) = 12 - 4 > 0 \Rightarrow \text{min}$

$f''(0) = 0 - 4 < 0 \Rightarrow \text{mal}$

$f''(1) = 12 - 4 > 0 \Rightarrow \text{min}$

The intervals of x-values which the curve is concave up $\left(-\infty, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$ concave down $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

x	$f(x)$
–2	8
–1	–1
–0.6	–0.6
0	0
0.6	0.6
1	–1



$\left(\textbf{Ans.} \, \boldsymbol{up}\left(-\infty - \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \infty\right); \boldsymbol{down}\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right)$

PROBLEM 4.17

Sketch the following curve by using the second derivative:

(1) $\boldsymbol{y = \frac{x}{1 + x^2}}$, **(2)** $\boldsymbol{y = -x(x - 7)^2}$, **(3)** $\boldsymbol{y = (x + 2)^2(x - 3)}$, **(4)** $\boldsymbol{y = x^2(5 - x)}$

Solution:

(1) $y = \frac{x}{1 + x^2} \Rightarrow \frac{dy}{dx} = \frac{(1 + x^2) \cdot 1 - x \cdot (2x)}{(1 + x^2)^2}$

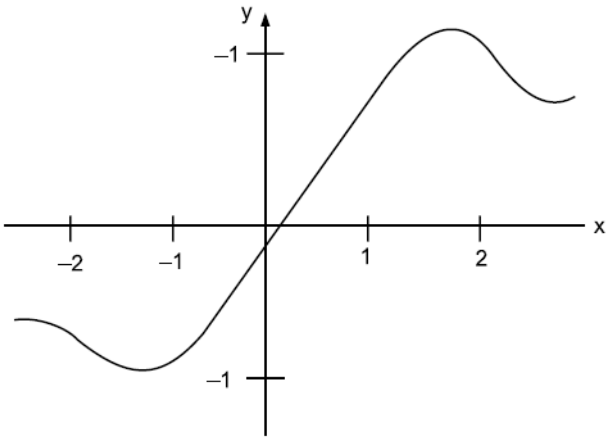
$= \frac{1 + x^2 - 2x^2}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2} = 0 \Rightarrow x = \pm 1$

$$\frac{d^2y}{dx^2}=\frac{(1+x^2)^2(-2x)-(1-x^2)2(1+x^2)\cdot 2x}{(1+x^2)^4}=\frac{(1+x^2)^2(-2x)-(1-x^2)2(1+x^2)\cdot 4x}{(1+x^2)^4}$$

$$=\frac{-2x-2x^3-4x+4x^3}{(1+x^2)^3}=\frac{-6x+2x^3}{(1+x^2)^3}=\frac{2x(x^2-3)}{(1+x^2)^3}$$

At $\quad x=-1\rightarrow\frac{d^2y}{dx^2}=\frac{-2(1-3)}{(1+1)^3}>0\Rightarrow \text{min}$

x	$f(x)$
–2	0.4
–1	–0.5
0	0
1	0.5
2	0.4



(Ans. max. (1, 0.5); min. (–1, – 0.5))

(2) $\quad y=-x(x-7)^2\Rightarrow y=-x(x^2-14x+49)$

$\Rightarrow \quad y'=-x(2x-14)+(x^2-14x+49)(-1)$

$$=-2x^2+14x-x^2+14x-49=-3x^2+28x-49=0$$

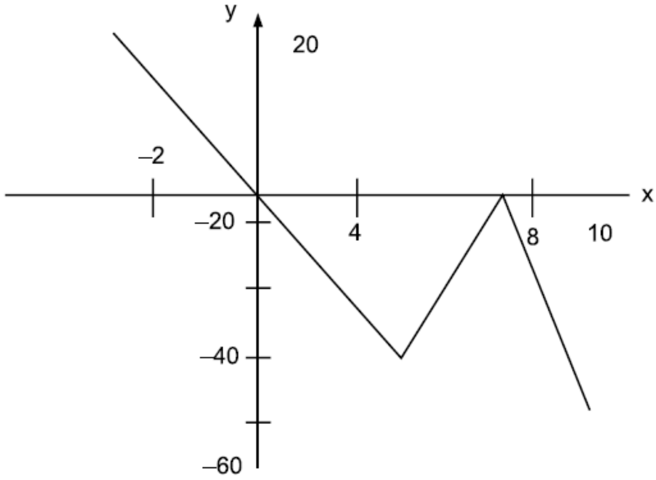
$\Rightarrow \quad -(x-7)(3x-7)=0\Rightarrow x=\frac{7}{3},7$

$$y''=-6x+28$$

At $\quad x=\frac{7}{3}\Rightarrow y''=-9\left(\frac{7}{3}\right)+28>0\Rightarrow \text{min}$

At $\quad x=7\Rightarrow y''=-6(7)+28<0\Rightarrow \text{max}$

x	y
0	0
7	–50.8
3	–36
4	0.5
0	7
10	–90



(Ans. max(7, 0); min(2.3, –50.8))

(3) $y = (x+2)^2(x-3)$

$$\Rightarrow \quad y' = (x+2)^2(1) + (x-3)2(x-2) = (x^2+4x+4) + (2x-6)(x-2)$$

$$= x^2 + 4x + 4 + 2x^2 + 4x - 6x - 12$$

$$= 3x^2 + 2x - 8$$

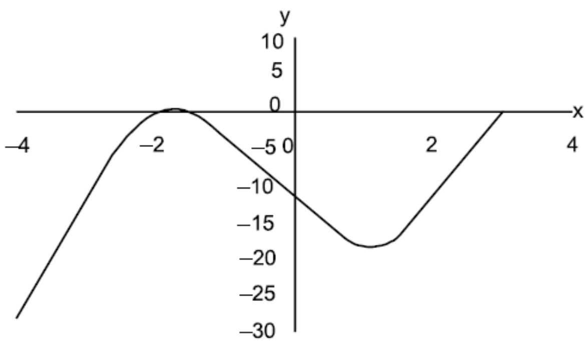
$$\Rightarrow (x+2)(3x-4) = 0$$

$$x = -2, \frac{4}{3}$$

$$y'' = (6x+2) \Rightarrow \text{At } x = -2, \quad y'' = 6(-2) + 2 < 0$$

At $x = \dfrac{4}{3} \quad \Rightarrow y'' = 6\left(\dfrac{4}{3}\right) + 2 > 0$

x	–4	–2	0	4/3	3
y	–28	0	–12	–18	0



(Ans. max. (–2, 0); min(1.3, –18.5))

(4) $y = 5x^2 - x^3$

$$\Rightarrow \qquad y' = 10x - 3x^2 = 0$$

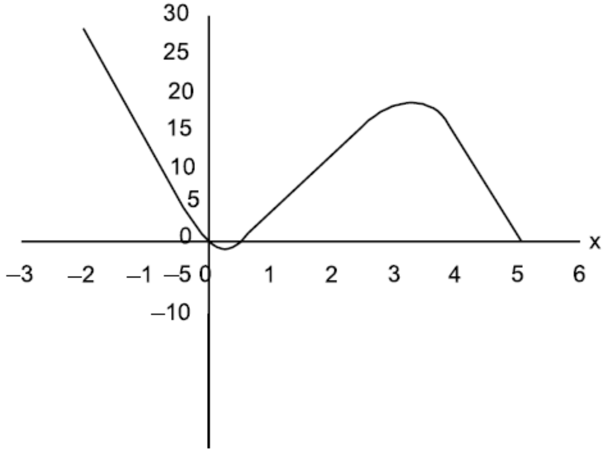
$$\Rightarrow \qquad x(10 - 3x) = 0$$

$$\Rightarrow \qquad x = 0, \frac{10}{3}$$

$$y'' = 10 - 6x \Rightarrow \text{at } x = 0 \Rightarrow y'' = 10 - 0 > \Rightarrow \text{min}$$

$$y'' = 10 - 6x \Rightarrow \text{at } x = \frac{10}{3} \Rightarrow y'' = 10 - 6\left(\frac{10}{3}\right) < 0 \Rightarrow \text{max}$$

x	–2	0	1	10/3	5
y	28	0	4	18.5	0



(Ans. max(3.3, 18.5); min(0, 0))

PROBLEM 4.18

What is the smallest perimeter possible for a rectangle of area 16 in^2?

Solution:

Area $= 16$

L is the length of the rectangle.

W is the width of the rectangle.

Then the area is $A = L \times w = 16 \Rightarrow L = \dfrac{16}{w}$

The perimeter is $P = 2(L+w) = 2\left(\frac{16}{w} + w\right)$

$$\frac{dp}{dw} = 2\left(\frac{-16}{w^2} + 1\right) = 0 \Rightarrow W = \pm 4 \text{ and } L = \frac{16}{\pm 4} = \pm 4$$

$$\frac{d^2p}{dw} = \frac{64}{w^3} \Rightarrow At\,W = \frac{d^2p}{dw^3} = \frac{64}{(4)^3} = 1 > 0 \,\text{Min}$$

$$\frac{d^2p}{dw} = \frac{64}{w^3} \Rightarrow at\,W = -4 \Rightarrow \frac{d^2p}{dw^3} = \frac{64}{(-4)^3} = -1 < 0 \Rightarrow \text{Max}$$

$\therefore$ The smallest perimeter is $P = 2(L+w) = 2(4+4) = 16\,\text{in}$.

(Ans. 16)

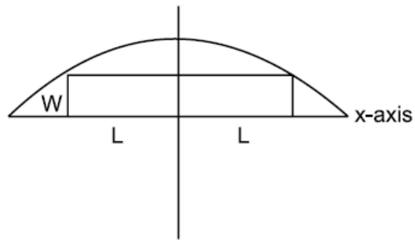
PROBLEM 4.19

Find the area of the largest rectangle with its lower base on the x-axis and upper vertices on the parabola $y = 12 - x^2$

Solution: The length of the rectangle is $2L$

The width is W, and the area of $\Rightarrow A = 2L * W$.

The parabola is $W = 12 - L^2$.



$$A = 2L(12 - L^2) = 24L - 2L^3$$

$$\frac{dA}{dL} = 24 - 6L^2 = 0 \Rightarrow L = \pm 2$$

and $$W = 12 - (\pm 2)^2 = 8$$

$$\frac{d^2A}{dL^2} = -12L \Rightarrow \text{at}\, L = 2 \Rightarrow \frac{d^2A}{dL^2} = -12 \times 2 = -24 < 0 \Rightarrow \text{Max}$$

$$\frac{d^2A}{dL^2} = -12L \Rightarrow \text{at}\, L = -2 \Rightarrow \frac{L^2}{16} = -12 \times (-2) = 24 > 0 \Rightarrow \text{Min}$$

$\therefore$ The area of the largest rectangle is $A = 2 \times 2 \times 8 = 32$.

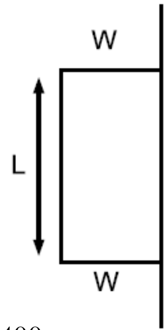
(Ans. 32)

PROBLEM 4.20

A rectangular plot is to be bounded on one side by a straight river and enclosed on the other three sides by a fence. You have $800\,\text{m}$ of fence at your disposal. What is the largest area you can enclose?

Solution:

W is the width of the plot.
L is the length of the plot.
The length of the fence is



$$2W + L = 800$$

$$\therefore \quad L = 800 - 2W$$

Then, area $\quad A = L \times W \Rightarrow W(800 - 2W)$

$$= 800W - 2W^2$$

$$\frac{dA}{dw} = 800 - 4W = 0 \Rightarrow W = 200 \text{ and } L = 800 - 400 = 400$$

$$\frac{d^2A}{dW^2} = -4 < 0 \, \text{Max}$$

$\therefore$ The largest area is $A = 200 * 400 = 80000\, m^2$.

(Ans. 80000)

PROBLEM 4.21

Show that the rectangle that has the maximum area for a given perimeter is a square.

Solution:

L is the length and w is the width of the rectangle.

The perimeter is $P = 2(L + w) \Rightarrow L = \frac{P}{2} - w$.

The area is $A = L \times w \Rightarrow A = \frac{P}{2} w - w^2$.

$$\frac{dA}{dw} = \frac{P}{2} - 2w = 0 \Rightarrow w = \frac{P}{4} \text{ and } L = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$$

$$\frac{d^2A}{dw^2} = -2 < 0 \Rightarrow \text{Max}$$

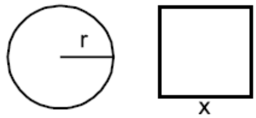
$\therefore$ The maximum area of the rectangle of a given perimeter P is a square where

$$L = w = \frac{P}{4}.$$

PROBLEM 4.22

A wire of length L is available for making a circle and a square. How should the wire be divided between the two shapes to maximize the sum of the enclosed areas?

Solution: r is the radius of the circle.
x is the length of the square.



The wire length is $L = 2\pi r + 4x \Rightarrow x = \frac{1}{4}(L - 2\pi r)$.

The total area is $A = \pi r^2 + x^2 \Rightarrow A = \pi r^2 + \frac{1}{16}(L - 2\pi r)^2$.

$$\frac{dA}{dr} = 2\pi r - \frac{\pi}{4}(L - 2\pi r) = 0 \Rightarrow r = \frac{L}{8 + 2\pi}$$

$$\frac{d^2A}{dr^2} = 2\pi - \frac{\pi}{4}(-2\pi) = 2\pi + \frac{\pi^2}{2} > 0 \Rightarrow \text{Min.}$$

Hence, the maximum value of A on the endpoints of the internal area is

$$0 \le 2\pi r \le L \Rightarrow 0 \le r \le \frac{L}{2\pi}$$

at $$r = 0 \Rightarrow x = \frac{L}{4} \Rightarrow A_1 = \frac{L^2}{16}$$

$$r = \frac{L}{2\pi} \Rightarrow x = 0 \Rightarrow A_2 = \frac{L^2}{4\pi}$$

Since $$A_2 = \frac{L^2}{4\pi} > A_1 = \frac{L^2}{16}$$

Hence, the wire should not be cut at all, but should be bent into a circle.

(Ans. All the wire should be bent into a circle.)

PROBLEM 4.23

A closed container is made from a right circular cylinder of radius r and height h with a hemispherical dome on top. Find the relationship between r and h that maximizes the volume for a given surface area s.

Solution:

$$V = \pi r^2 h + \frac{2}{3} r^3 \pi$$

$$V = \pi r^2 \times \frac{s - 6r^2\pi}{2r\pi} + \frac{2}{3} r^3 \pi$$

$$V = \frac{1}{2} sr - 3r^3\pi + \frac{2}{3} r^3 \pi$$

$$V' = \frac{1}{2} s + \frac{1}{2} rs' - 9r^2\pi + 2r^2\pi$$

$$V' = \frac{1}{2} s + \frac{1}{2} rs' - 7r^2\pi$$

$$V' = \frac{1}{2} s + \frac{1}{2} r \times 18r\pi - 7r^2\pi$$

$$0 = \frac{1}{2} s + 9r^2\pi - 7r^2\pi$$

$$0 = 3r^2\pi + 2rh + 2r^2\pi$$

$$0 = 3r + 2h + 2r$$

$$5r + 2h = 0$$

$$s = 4r^2\pi$$

$$\frac{s}{4\pi} = r^2$$

$$r = \frac{\sqrt{3}}{2\sqrt{\pi}}$$

$$h = \frac{s - 4r^2\pi}{2r\pi} = \frac{s - (4)^2 \times \frac{s}{\pi} \times \pi}{2 \times \frac{\sqrt{s}}{2\sqrt{\pi}} \times \pi}$$

$$h = \frac{s - 2s}{\sqrt{s} * \sqrt{\pi}} = \frac{-s}{\sqrt{s} * \sqrt{\pi}}$$

$$h = \frac{-\sqrt{s}}{\sqrt{\pi}}$$

(Ans. $\boldsymbol{r = h = \sqrt{\frac{s}{5\pi}}}$ **)**

PROBLEM 4.24

An open rectangular box is to be made from a piece of cardboard 8 in wide and 15 in long by cutting a square from each corner and bending up the sides. Find the dimensions of the box of largest volume.

Solution: L is the length and W is the width and H the height of the rectangular box.
The length of the cutting a square is x.
$\therefore$ The length $= 8 - 2x$.
The width is $= 15 - 2x$.
The height is $= x$.

$$V = L \times W \times H$$
$$V = (8 - 2x)(15 - 2x)x$$
$$V = (120 - 16x - 30x + 4x^2)x$$
$$V = (120 - 46x + 4x^2)x$$
$$V = 120x - 46x^2 + 4x^3$$
$$\frac{dV}{dx} = 120 - 92x + 12x^2 \Rightarrow \frac{dV}{dx} = 0 \Rightarrow [0 = 120 - 92x + 12x^2] \div 4$$
$$30 - 23x + 3x^2 = 0$$
$$3x^2 - 23x + 30 = 0$$
$$(3x - 15)(x - 2) = 0$$
$$3x - 15 = 0 \Rightarrow 3x = 15 \Rightarrow x = \frac{15}{3} \Rightarrow x = 5$$
$$x - 2 = 0 \Rightarrow x = 2$$
$$V = (8 - 2(2))(15 - 2(2))^2$$
$$V = 4 \times 11 \times 2 = 88\, m^3$$

(Ans. height = 5/3; width = 14/3; length = 35/3)

CHAPTER 5

INTEGRATION

PROBLEMS

PROBLEM 5.1

Evaluate $\int (x^2 - 1) \cdot (4 - x^2) dx$

Solution: $\int (x^2 - 1) \cdot (4 - x^2) dx$

$$\int (4x^2 - x^4 - 4 + x^2) dx$$

$$\int (5x^2 - x^4 - 4) dx$$

$$\frac{5x^3}{3} - \frac{x^5}{5} - 4x + c \qquad \left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$$

(Ans. $\frac{5}{3}x^3 - \frac{1}{5}x^5 - 4x + c$**)**

PROBLEM 5.2

Evaluate $\int e^x \cdot \sin e^x dx$

Solution: $\int e^x \cdot \sin e^x dx \Rightarrow \int \sin u \cdot du = -\cos u + c$

$$\therefore \quad \int e^x \cdot \sin e^x dx = -\cos e^x + c$$

(Ans. $-\cos e^x + c$**)**

PROBLEM 5.3

Evaluate $\int \tan(3x+5)dx$

Solution: $\int \tan(3x+5)dx$

$$\left[\because \int \tan u \cdot du = -\ln|\cos u| + c\right]$$

$$= \frac{1}{3}\int 3\tan(3x+5)dx$$

$$= -\frac{1}{3}\ln|\cos(3x+5)| + c$$

(*Ans.* $-\frac{1}{3}\ln|\cos(3x+5)|+c$)

PROBLEM 5.4

Evaluate $\int \frac{\cot(\ln x)}{x}dx$

Solution: $\int \frac{\cot(\ln x)}{x}dx$

$$\int \cot u \cdot du = \ln|\sin(\ln x)| + c$$

(Ans. $\ln|\sin(\ln x)|+c$)

PROBLEM 5.5

Evaluate $\int \frac{\sin x + \cos x}{\cos x}dx$

Solution: $\int \frac{\sin x + \cos x}{\cos x}dx$

$$\Rightarrow \quad \int\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\cos x}\right)dx$$

$$\int (\tan x + 1)dx = -\ln|\cos(x)| + x + c$$

(Ans. $\ln|\cos x| + x + c$)

PROBLEM 5.6

Evaluate $\int \frac{dx}{1+\cos x}$

Solution: $\int \frac{dx}{1+\cos x}$

$$\int \frac{dx}{1+\cos x} \times \frac{1-\cos x}{1-\cos x}$$

$$\Rightarrow \int \frac{1-\cos x}{\sin^2 x} dx = \int \left[\frac{1}{\sin^2 x} - \frac{\cos x}{\sin x} \right] dx$$

$$\Rightarrow [\csc^2 x - \cot x \cdot \csc x] dx$$

$$[\text{As} \int \csc^2 u du = -\cot u + \text{c}$$

$$\int \csc u \cdot \cot u \cdot du = -\csc u + c]$$

$$\therefore \int \frac{dx}{1+\cos x} = -\cot x + \csc x + c$$

(Ans. $-\cot x + \csc x + c$)

PROBLEM 5.7

Evaluate $\int \cot(2x+1) \cdot \csc^2(2x+1)\, dx$

Solution:

$$\int \cot(2x+1) \cdot \csc^2(2x+1)\, dx = -\frac{1}{2} \int \cot(2x+1)(-2\csc^2(2x+1)dx)$$

$$= -\frac{1}{2} \times \frac{\cot^2(2x+1)}{2} + c = -\frac{1}{4} \times \cot^2(2x+1) + c$$

$$[\text{As} \frac{d}{dx}(2x+1) = 2$$

$$\int (f(x))^n f'(x) dx = \frac{f(x)^{n+1}}{n+1}]$$

(Ans. $-\frac{1}{4}\cot^2(2x+1) + c$)

PROBLEM 5.8

Evaluate $\int \frac{dx}{\sqrt{1-9x^2}}$

Solution:

$$\int \frac{1}{\sqrt{a-u^2}} \cdot du \Rightarrow \sin^{-1}\frac{u}{a} + C$$

$$\therefore \qquad \int \sqrt{1-(3x)^2}\,dx$$

$$= \frac{1}{3}\int \frac{3dx}{\sqrt{1-(3x)^2}}$$

$$\left(\text{As}\,\frac{d}{dx}(3x) = 3\right)$$

$$\therefore \qquad \int \frac{dx}{\sqrt{1-9x^2}} = \frac{1}{3}\sin^{-1}(3x) + c$$

(Ans. $\boldsymbol{\frac{1}{3}\sin^{-1}(3x)+c}$**)**

PROBLEM 5.9

Evaluate $\boldsymbol{\int \frac{dx}{\sqrt{2-x^2}}}$

Solution: $\int \frac{dx}{\sqrt{2-x^2}}dx$ Here, $u = x, a = \sqrt{2}$

$$\therefore \qquad \int \frac{dx}{\sqrt{\left(\sqrt{2}\right)^2 - (x)^2}} = \sin^{-1}\frac{x}{\sqrt{2}} + c$$

(Ans. $\boldsymbol{\sin^{-1}\frac{x}{\sqrt{2}}+c}$**)**

PROBLEM 5.10

Evaluate $\boldsymbol{\int e^{2x}\cdot\cosh e^{2x}dx}$

Solution: $\int e^{2x}\cosh e^{2x}dx$

Here, $u = e^{2x}, du = 2e^{2x}$

$$\therefore \qquad \int e^{2x}\cosh e^{2x}dx = \frac{1}{2}\int (2e^{2x})\cosh e^{2x}dx = \frac{1}{2}\sinh e^{2x} + c$$

(Ans. $\boldsymbol{\frac{1}{2}\sinh e^{2x}+c}$**)**

PROBLEM 5.11

Evaluate $\int e^{\sin x} \cdot \cos x\, dx$

Solution: $\int e^{\sin x} \cos x\, dx$

As, $\int e^u du = e^u + c$

and $u = \sin x$

$du = \cos x$

$\therefore \quad \int e^{\sin x} \cos x\, du = e^{\sin x} + c$

(Ans. $e^{\sin x} + c$**)**

PROBLEM 5.12

Evaluate $\int \frac{dx}{e^{3x}}$

Solution:

$$\int \frac{dx}{e^{3x}} = \int e^{-3x} dx$$

$$\int e^u \cdot du = e^u + c$$

$$u = -3x, du = -3$$

$\Rightarrow \quad -\frac{1}{3}\int e^{-3x}(-3\,dx) = -\frac{1}{3}e^{-3x}$

(Ans. $-\frac{1}{3}e^{3x} + c$**)**

PROBLEM 5.13

Evaluate $\int \frac{e^{\sqrt{x}} - 1}{\sqrt{x}} dx$

Solution:

$$\int \frac{e^{\sqrt{x}} - 1}{\sqrt{x}} dx \Rightarrow \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx - \int \frac{1}{\sqrt{x}} dx$$

$$\int e^u = du = e^u + c$$

$$u = \sqrt{x}$$

$$du = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \quad 2\int e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}dx\right)-\int\frac{1}{\sqrt{x}}dx$$

$$\sqrt{x}=x^{-\frac{1}{2}},\ \frac{1}{\sqrt{x}}=x^{-\frac{1}{2}}$$

$$\int x^{-\frac{1}{2}}=\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}=\frac{x^{\frac{1}{2}}}{\frac{1}{2}}$$

$$\Rightarrow \quad \int e^{\sqrt{x}}\left(\frac{dx}{2\sqrt{x}}\right)-\int x^{-\frac{1}{2}}dx$$

$$=2e^{\sqrt{x}}-\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+c$$

$$=2e^{\sqrt{x}}-2\sqrt{x}+c$$

(Ans. $\mathbf{2e^{\sqrt{x}}-2\sqrt{x}+c}$**)**

PROBLEM 5.14

Evaluate $\int x\left(a+b\sqrt{3x}\right)dx$**, where** a **and** b **are constants.**

Solution:

$$\int x(a+b\sqrt{3x})dx=\int (ax+b\sqrt{3}\sqrt{x}\cdot x)dx \quad \sqrt{x}=x^{\frac{1}{2}},\ x^{\frac{1}{2}}-x^{1}=x^{\frac{3}{2}}$$

$$=\int (ax+b\sqrt{3}x^{\frac{3}{2}})dx$$

$$=\frac{ax^2}{2}+b\sqrt{3}\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+c=\left[\frac{ax^2}{2}+\frac{2b\sqrt{3}x^{\frac{5}{2}}}{5}\right]+c$$

$$=\frac{1}{10}\left(5ax^2+b4\sqrt{3}x^{\frac{5}{2}}\right)+c$$

(Ans. $\mathbf{\frac{1}{10}\left(5ax^2+b4\sqrt{3x}^{\frac{5}{2}}\right)+c}$**)**

PROBLEM 5.15

Evaluate $\displaystyle\int \frac{dx}{-1-x^2}$

Solution: $\displaystyle\int \frac{dx}{-1-x^2} \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}+c$

$$= \int \frac{-dx}{1+x^2} = -\int \frac{dx}{1+x^2}\, u = x, du = 1, a = 1$$

$\therefore \quad -\tan^{-1} x + c$

(Ans. $-\tan^{-1} x + c$)

PROBLEM 5.16

Evaluate $\displaystyle\int \frac{\cos\theta\, d\theta}{1+\sin^2\theta}$

Solution:

$$\int \frac{\cos\theta\, d\theta}{1+\sin^2\theta} = \int \frac{1}{1+(\sin\theta)^2}(\cos\theta)d\theta$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}+c$$

$$u = \sin\theta\, du = \cos\theta\, a = 1$$

$\therefore \quad \displaystyle\int \frac{\cos\theta}{1+(\sin\theta)^2} d\theta$

$$= \tan^{-1}(\sin\theta) + c$$

(Ans. $\tan^{-1}(\sin\theta) + c$)

PROBLEM 5.17

Evaluate $\displaystyle\int \frac{1}{x^2}\csc\frac{1}{x}\cot\frac{1}{x}dx$

Solution:

$$\int \frac{1}{x^2}\csc\frac{1}{x}\cdot\cot\frac{1}{x}dx = \int \csc\frac{1}{x}\cdot\cot\frac{1}{x}\left(\frac{1}{x^2}\right)dx$$

$$= -\int \csc\frac{1}{x}\cdot\cot\frac{1}{x}\left(-\frac{1}{x^2}\right)dx$$

$$= -\left(-\csc\frac{1}{x}+c\right) = \csc\frac{1}{x}+c$$

$$[\because \int \csc u \cot u \cdot du = -\csc u + c]$$

$$\frac{d}{dx}\frac{1}{x} = \frac{x\times 0-1}{x^2}\times -\frac{1}{x^2}$$

(Ans. $\csc\frac{1}{x}+c$)

PROBLEM 5.18

Evaluate $\int \frac{3x+1}{\sqrt[3]{3x^2+2x+1}}dx$

Solution: $\int \frac{3x+1}{\sqrt[3]{3x^2+2x+1}}dx$

$$\Rightarrow \int \frac{(3x+1)}{(3x^2+2x+1)^{\frac{1}{3}}}dx = \int (3x^2+2x+1)^{\frac{1}{3}}(3x+1)dx$$

$$= \frac{1}{2}\int (3x^2+2x+1)^{-\frac{1}{3}}(6x+2)dx$$

$$= \frac{1}{2}\frac{(3x^2+2x+1)^{\frac{2}{3}}}{\frac{2}{3}} \qquad \int f(x)f'(x)dx = \frac{f(x)^{n+1}}{n+1}$$

$$= \frac{3}{4}\sqrt[3]{(3x^2+2x+1)^2}+c$$

(Ans. $\frac{3}{4}3\sqrt{(3x^2+2x+1)^2}+c$)

PROBLEM 5.19

Evaluate $\int \sin(\tan\theta)\cdot\sec^2\theta\, d\theta$

Solution:

$$\int \sin(\tan\theta)\cdot \sec^2\theta\, d\theta$$

$$\int \sin u . du = -\cos u + c \text{ and } u = \tan\theta\, du = \sec^2\theta$$

$$\therefore \quad \int \sin(\tan\theta)\cdot\sec^2\theta\, d\theta = -\cos(\tan\theta) + c$$

(Ans. $-\cos(\tan\theta)+c$)

PROBLEM 5.20

Evaluate $\int \sqrt{x^2 - x^4}\,dx$

Solution:

$$\int \sqrt{x^2 - x^4}\,dx = \int \sqrt{x^2(1-x^2)}\,dx$$

$$= \int x^2(1-x^2)^{\frac{1}{2}}\,dx = \int (x^2)^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}$$

$$= -\frac{1}{2}\int -2x\sqrt{(1-x^2)}\,dx = \frac{\frac{1}{2}(1-x^2)^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= -\frac{1}{3}\sqrt{(1-x^2)^3} + c$$

(Ans. $-\frac{1}{3}\sqrt{(1-x^2)^3}+c$)

PROBLEM 5.21

Evaluate $\int \frac{\sec^2 2x dx}{\sqrt{\tan 2x}}$

Solution:

$$\int \frac{\sec^2 2x dx}{\sqrt{\tan 2x}} = \int \sec^2(2x)\cdot(\tan 2x)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{2}\int 2\sec^2(2x)\cdot(\tan 2x)^{-\frac{1}{2}}\,dx$$

$$= \frac{1}{2}\cdot\frac{(\tan 2x)^{\frac{1}{2}}}{1/2} + c$$

$$= \sqrt{\tan 2x} + c$$

PROBLEM 5.22

Evaluate $\int (\sin\theta - \cos\theta)^2\, d\theta$

Solution: $\int (\sin\theta - \cos\theta)^2\, d\theta$

$\Rightarrow$ $\int (\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta)\, d\theta$

As $\sin^2\theta + \cos^2\theta = 1$

$$\therefore \quad = \int (1 - 2\sin\theta\cos\theta)\, d\theta$$

$$= \theta + 2\frac{\cos^2\theta}{2} + c = \theta + \cos^2\theta + c$$

$$[\because \quad \int f(x) f'(x) dx = \frac{f(x)^{n+1}}{n+1}$$

$$f(x) = \cos\theta$$

$$f'(x) = -\sin\theta - (2\cos\theta\sin\theta)$$

$$= 2\cos\theta(-\sin\theta)]$$

(Ans. $\boldsymbol{\theta + 2\frac{\cos^2\theta}{2} + c = \theta + \cos^2\theta + c}$**)**

PROBLEM 5.23

Evaluate $\int \frac{y}{y^4 + 1}\, dy$

Solution:

$$= \int \frac{y}{y^4 + 1} \cdot y\, dy$$

$$= \int \frac{y}{(y^2)^2 + 1} \cdot y\, dy \qquad \left[\because \int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c\right]$$

$$a = 1 \; u = y^2 \; du = 2y$$

$$= \frac{1}{2}\int \frac{1}{(y^2)^2 + 1} 2y\, dy$$

$$= \frac{1}{2}\tan^{-1} y^2 + c$$

(Ans. $\boldsymbol{\frac{1}{2}\tan^{-1} y^2 + c}$**)**

PROBLEM 5.24

Evaluate $\int \frac{dx}{\sqrt{x}(x+1)}$

Solution:

$$\int \frac{dx}{\sqrt{x}(x+1)} \Rightarrow \int \frac{1}{(x+1)} \cdot \frac{1}{\sqrt{x}} dx$$

$$\Rightarrow \quad \int \frac{1}{(\sqrt{x})^2+1} \cdot \frac{1}{\sqrt{x}} dx = \int \frac{1}{\left(1+(\sqrt{x})^2\right.} \cdot \frac{1}{\sqrt{x}} dx$$

$$[\text{Here, } a=1 \quad u=\sqrt{x} \quad du = \frac{1}{2}(x)^{-\frac{1}{2}}]$$

$$= 2\int \frac{1}{1+\left(\sqrt{x}\right)^2} \cdot \frac{1}{2\sqrt{x}} dx$$

$$= 2\tan^{-1}(\sqrt{x}) + c$$

(Ans. $\mathbf{2\tan^{-1}\sqrt{x}+c}$**)**

PROBLEM 5.25

Evaluate $\int t^{\frac{2}{3}}(t^{\frac{5}{3}}+1)^{\frac{2}{3}} dt$

Solution:

$$f(x) = \left(t^{\frac{5}{3}}+1\right)$$

$$f'(x) = \frac{5}{3}t^{\frac{2}{3}}$$

&

$$\int f(x) f'(x) dx = \frac{f(x)^{n+1}}{n+1} + c$$

$$\int t^{\frac{2}{3}}(t^{\frac{5}{3}}+1)^{\frac{2}{3}} dt$$

$$\frac{3}{5}\int \frac{5}{3} t^{\frac{2}{3}}(t^{\frac{5}{3}}+1)^{\frac{2}{3}} dt \quad = \frac{3}{5}\frac{(t^{\frac{5}{3}}+1)^{\frac{5}{3}}}{\frac{5}{3}} + c$$

$$= \frac{9}{25}(t^{\frac{5}{3}}+1)^{\frac{5}{3}} + c$$

(Ans. $\mathbf{\frac{9}{25}(t^{\frac{5}{3}}+1)^{\frac{5}{3}}+c}$**)**

PROBLEM 5.26

Evaluate $\displaystyle\int \frac{dx}{x^{\frac{1}{5}}\sqrt{1+x^{\frac{4}{5}}}}$

Solution: $\displaystyle\int x^{-\frac{1}{5}}(1+x^{\frac{4}{5}})^{-\frac{1}{2}}dx$

$$= \frac{5}{4}\left(\int \frac{4}{5}x^{-\frac{1}{5}}(1+x^{\frac{4}{5}}(1+x^{\frac{4}{5}})\right)^{\frac{1}{2}} dx$$

$$= \frac{5}{4}\frac{(1+x^{\frac{4}{5}})^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{10}{4}(1+x^{\frac{4}{5}})^{\frac{1}{2}} + c$$

$$= \frac{5}{2}\sqrt{1+x^{\frac{4}{5}}} + c$$

(Ans. $\boldsymbol{\frac{5}{2}\sqrt{1+x^{\frac{4}{5}}} + c}$**)**

PROBLEM 5.27

Evaluate $\displaystyle\int \frac{(\cos^{-1} 4x)^2}{\sqrt{1-16x^2}}dx$

Solution: $\displaystyle\int \frac{(\cos^{-1} 4x)^2}{\sqrt{1-16x^2}}dx$

$$\Rightarrow \int (\cos^{-1} 4x)^2 \cdot \frac{1}{\sqrt{1-16x^2}}dx$$

Since, $\displaystyle\frac{d}{dx}\cos^{-1} u = -\frac{1}{\sqrt{1-u^2}}du$

Let, $f(x) = (\cos^{-1} 4x)^2$

$\therefore$ $\displaystyle f'(x) = \cos^{-1} 4x\left(-\frac{1}{\sqrt{1-16x^2}}\right)$

$$\therefore \qquad -\frac{1}{4}\int (\cos^{-1} 4x)^2 \frac{-4}{\sqrt{1-(4x)^2}}dx$$

$$= -\frac{1}{4}\int (\cos^{-1} 4x)^2 \frac{-4}{\sqrt{1-(4x)^2}}dx$$

$$= -\frac{1}{4}\frac{(\cos^{-1} 4x)^3}{3} + c \quad \Rightarrow \quad -\frac{1}{12}(\cos^{-1} 4x)^3 + c$$

(Ans. $-\frac{1}{12}(\cos^{-1} 4x)^3 + c$)

PROBLEM 5.28

Evaluate $\int \frac{dx}{x\sqrt{4x^2-1}}$

Solution: $\int \frac{dx}{x\sqrt{4x^2-1}}dx$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + c$$

$$u = 2x\, du = 2$$

$$\int \frac{dx}{x\sqrt{(2x)^2-1}} \quad \Rightarrow \quad 2\int \frac{dx}{2x\sqrt{(2x)^2-1}}$$

$$= \sec^{-1}(2x) + c$$

(Ans. $\sec^{-1}(2x) + c$)

PROBLEM 5.29

Evaluate $\int \frac{dx}{(e^x + e^{-x})^2}$

Solution: $\int \frac{dx}{(e^x + e^{-x})^2}$

$$\operatorname{sech} u = \frac{1}{\cosh} = \frac{2}{e^u + e^{-u}}$$

$$= \frac{1}{4}\int \left(\frac{2}{e^x + e^{-x}}\right)^2 dx = \frac{1}{4}\int \operatorname{sech}^2 x dx$$

$$= \frac{1}{4}\int \tan x + c$$

$$\int \operatorname{sech}^2 u\, du = \tanh u + c$$

(Ans. $\frac{1}{4}\tanh x + c$)

PROBLEM 5.30

Evaluate $\int 3^{\ln x^2} \frac{dx}{x}$

Solution:

$$\int 3^{\ln x^2} \frac{dx}{x} = \frac{1}{2}\int 3^{\ln x^2} \frac{2dx}{x}$$
$$= \frac{1}{2} e^{\ln x^2} \frac{1}{\ln 3} + c$$
$$= \frac{1}{2\ln 3} 3^{\ln x^2} + c$$

$$a^u du = \frac{a^u}{\ln a} + c$$
$$a = 3 \quad u = \ln x^2$$
$$\int du = \frac{1}{x^2}$$
$$= \frac{2x}{x^2}$$

(Ans. $\frac{1}{2\ln 3} 3^{\ln x^2} + c$**)**

PROBLEM 5.31

Evaluate $\int \frac{\cot x\, dx}{\ln(\sin x)}$

Solution:

$$\int \frac{\cot x}{\ln(\sin x)} dx = \int \frac{1}{\ln(\sin x)} \cdot \cot x\, dx$$
$$= \ln(\ln \sin x) + c$$

[since $\int \frac{1}{u} du = \ln u + c$

$u = \ln(\sin x)$]

(Ans. $(\ln \sin x) + c$**)**

PROBLEM 5.32

Evaluate $\int \frac{(\ln x)^2}{x} dx$

Solution:

$$\int \frac{(\ln x)^2}{x}dx = \int (\ln x)^2 \cdot \frac{1}{x}dx$$

$$f(x) = (\ln x)^2$$

$$f'(x) = \frac{1}{x}$$

$$= \frac{(\ln x)^3}{3} + c = \frac{1}{3}(\ln x)^3 + c$$

(Ans. $\frac{1}{3}(\ln x)^3 + c$)

PROBLEM 5.33

Evaluate $\int \frac{\sin x \cdot e^{\sec x}}{\cos^2 x}dx$

Solution:

$$\int \frac{\sin x \cdot e^{\sec x}}{\cos^2 x}dx = \int \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}e^{\sec x}dx$$

$$= \int e^{\sec x}(\sec x \cdot \tan x)\frac{d}{dx}\sec x^2 + \sec x \tan x$$

$$\int e^{su}du = e^u + c$$

$$u = \sec x$$

$$du = \sec x + \tan x$$

(Ans. $e^{\sec x} + c$)

PROBLEM 5.34

Evaluate $\int \frac{dx}{x \cdot \ln x}$

Solution: $\int \frac{dx}{x \ln x} \quad \Rightarrow \quad \int \frac{dx}{x} \cdot \frac{1}{\ln x}$

$$= \ln(\ln x) + c$$

[Since $\int \frac{1}{u}du = \ln u + c$

$$u = \ln x$$

$du = \frac{1}{x}$]

(Ans. $\frac{1}{3}(\ln x)^3 + c$)

PROBLEM 5.35

Evaluate $\int \frac{d\theta}{\cosh\theta + \sinh\theta}$

Solution:

$$\int \frac{d\theta}{\cosh\theta + \sinh\theta}; \quad \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}, \quad \cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

$$\Rightarrow \int \frac{d\theta}{\frac{e^{\theta} + e^{-\theta}}{2} + \frac{e^{\theta} - e^{-\theta}}{2}} = \int \frac{d\theta}{\frac{e^{\theta} - e^{-\theta} + e^{\theta} + e^{-\theta}}{2}} = 2\int \frac{1}{2e^{\theta}} d\theta = \int e^{-\theta} d\theta$$

$$= -e^{-\theta} + c$$

(Ans. $-e^{-\theta} + c$**)**

PROBLEM 5.36

Evaluate $\int \frac{2^x - 8^{2x}}{\sqrt{4^x}} dx$

Solution:

$$\int \frac{2^x - 8^{2x}}{\sqrt{4^x}} dx$$

$$= \int \frac{2^x - 2^{6x}}{2^x} dx = \int (1 - 2^{5x}) dx$$

$$= \int dx - \int 2^{5x} dx = x - \frac{1}{5}\int 2^{5x} 5dx$$

$$= x - \frac{1}{5}\frac{2^{5x}}{\ln 2} + c$$

$$= x - \frac{1}{5}\frac{2^{5x}}{\ln 2} + c$$

$\sqrt{u^x} = 2^x$

$\sqrt{u^2} = 2^2$

$8^{2x} = 2^{3(2x)} = 2^{6x}$

$\int a^u du$

$= \frac{a^u}{\ln a} + c$

$u = 5x$

$du = 5$

(Ans. $x - \frac{1}{5\ln 2} e^{5x} + c$**)**

PROBLEM 5.37

Evaluate $\int \frac{e^{\tan^{-1} 2t}}{1 + 4t^2} dt$

Solution:

$$\int \frac{e^{\tan^{-1} 2t}}{1+4t^2} dt$$

$$\Rightarrow \quad \int e^{\tan^{-1} 2t} \frac{1}{1+4t^2} dt$$

$$\Rightarrow \quad \int e^{\tan^{-1} 2t} \frac{1}{1+(2t)^2} dt$$

$$[\text{since} \int e^u \cdot du = e^u + c$$

$$u = \tan^{-1} 2t$$

$$du = \frac{1}{1+(2t)^2} \cdot 2]$$

$$\therefore \quad \int \frac{e^{\tan^{-1} 2t}}{1+4t^2} dt = \frac{1}{2} \int e^{\tan^{-1} 2t} \frac{2\,dt}{1+(2t)^2}$$

$$= \frac{1}{2} e^{\tan^{-1} 2t} + c$$

(Ans. $\frac{1}{2} e^{\tan^{-1} 2t} + c$**)**

PROBLEM 5.38

Evaluate $\int \frac{\cot x}{\csc x} dx$

Solution:

$$\int \frac{\cot x}{\csc x} dx = \int \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}} dx$$

$$\int \cos x\, dx = \sin x + c$$

(Ans. $\sin x + c$**)**

PROBLEM 5.39

Evalute $\int \sec^4 x \cdot \tan^3 x\, dx$

Solution:

$$\int \sec^4 x \cdot \tan^3 x\, dx$$

$$\sec^2 x = (\tan^2 x + 1)$$

$$\int \sec^2 x \cdot \sec^2 x \cdot \tan^3 x\,dx \quad \Rightarrow \quad \int \sec^2 x(1+\tan^2 x)\tan^2 x)\,dx$$

$$\Rightarrow \qquad \int [\sec^2 x(\tan^5 x + \tan^3 x)]\,dx$$

$$\Rightarrow \qquad \int \tan^5 x(\sec^3 x)dx + \int \tan^3 x(\sec^3 x\,dx)$$

$$\Rightarrow \qquad \frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} + c$$

$$\textbf{(Ans. } \frac{1}{6}\tan^6 + \frac{1}{4}\tan^4 x + c\textbf{)}$$

PROBLEM 5.40

Evaluate $\int \csc^4 3x\,dx$

Solution:

$$\int \csc^4 3x\,dx$$

$$\Rightarrow \qquad \int \csc^2 3x \cdot \csc^2 3x\,dx$$

$$\Rightarrow \qquad \int \csc^2 3x(\cot^2 3x + 1)\,dx$$

$$\Rightarrow \qquad \int \csc^2 3x \cdot \cot^2 3x\,dx + \int \csc^2 3x\,dx$$

$$[\text{since} \int \csc^2 u \cdot du = -\cot u + c$$

$$\int f(x) = \cot 3x$$

$$\int f'(x) = -\csc^2 3x \cdot 3]$$

$$\textbf{(Ans. } -\frac{1}{9}\cot^3 3x - \frac{1}{3}\cot 3x + c\textbf{)}$$

$$\therefore \quad \int \csc^4 3x\,dx = -\frac{1}{9}\cot^3 3x - \frac{1}{3}\cot 3x + c$$

PROBLEM 5.41

Evaluate $\int \frac{\cos^3 t}{\sin^2 t}\,dt$

Solution: $\int \frac{\cos^3 t}{\sin^2 t} \cdot dt$

$$\Rightarrow \qquad \int \frac{\cos t \cdot \cos^2 t}{\sin^2 t} \cdot dt$$

$$\Rightarrow \qquad \int \frac{\cos t(1-\sin^2 t)}{\sin^2 t} \cdot dt = \int \frac{\cos t}{\sin^2} \cdot dt - \int \cos t \cdot dt$$

$$= \int \frac{\cos t}{\sin t} \cdot \frac{1}{\sin t} \cdot dt - \int \cos t \cdot dt$$

$$= \int \cot t \cdot \csc t \cdot dt - \cos t \cdot dt$$

$$= -\text{cosec}\, t - \sin t + c$$

(Ans. $\mathbf{cosec}\, t - \sin t + c$)

PROBLEM 5.42

Evaluate $\int \frac{\sec^4 x}{\tan^4 x} dx$

Solution:

$$\int \frac{\sec^4 x}{\tan^4 x} \cdot dx = \int \frac{\left(\frac{1}{\cos^4 x}\right)}{\frac{\sin^4 x}{\cos^4 x}} \cdot dx = \int \frac{1}{\sin^4 x} \cdot dx$$

$$= \int \csc^4 x \cdot dx$$

$$= \int \csc^2 x \cdot (\csc^2 x) \cdot dx$$

$$= \int \csc^2 x \cdot (\cot^2 x + 1) \cdot dx \qquad (\because \csc^2 x = 1 + \cot^2 x)$$

$$(\because \frac{d}{dx} \cot x = -\csc^2 x)$$

$$= \int \csc^2 x \,.\, \cot^2 x \cdot dx + \int \csc^2 x \cdot dx$$

$$= -\int \csc^2 x \cdot \cot^2 x \cdot dx + \int \csc^2 x \cdot dx$$

$$= -\frac{\cot^3 x}{3} - \cot x + c$$

(Ans. $-\frac{1}{3} \cot^3 x - \cot x + c$)

PROBLEM 5.43

Evaluate $\int \tan^2 4\theta \cdot d\theta$

Solution:

$$\int \tan^2 4\theta \cdot d\theta$$

Since,

$$\tan^2 u = \sec^2 u - 1$$

$$= \int (\sec^2 4\theta - 1) \cdot d\theta$$

$$= \frac{1}{4} \int \sec^2 4\theta \cdot 4 \cdot d\theta - \int d\theta$$

$\because$
$$\int \sec^2 u.du = \tan u + c$$

$$u = 4\theta \quad du = 4$$

$$= \frac{1}{4} \tan 4\theta - \theta + c$$

(Ans. $\mathbf{\frac{1}{4}\tan 4\theta - \theta + c}$**)**

PROBLEM 5.44

Evaluate $\int \frac{e^x}{1+e^x} dx$

Solution: $\int \frac{e^x}{1+e^x} \cdot dx$

$\because$
$$\int \frac{1}{u} \cdot du = \ln u + c$$

$$u = 1 + e^x$$

$$du = e^x$$

$\Rightarrow$
$$\int \frac{1}{1+e^x} \cdot e^x \cdot dx$$

$$= \ln(1+e^x) + c$$

(Ans. $\mathbf{\ln(1+e^x) + c}$**)**

PROBLEM 5.45

Evaluate $\int \tan^3 2x \, dx$

Solution: $\int \tan^3 2x \cdot dx$

$$\tan^3 2x = \tan x \cdot \tan^2 2x$$

$$= \int \tan 2x \cdot \tan^2 2x \cdot dx$$

$$\tan^2 2x = (\sec^2 2x - 1)$$

$$= \int \tan 2x(\sec^2 2x - 1)dx$$

$$= \int \tan 2x \cdot \sec^2 2x \cdot dx - \int \tan 2x \cdot dx \left(\because \frac{d}{dx}\tan x = \sec^2 u \cdot \frac{du}{dx}\right)$$

$$= \frac{1}{2}\int \tan 2x \cdot \sec^2 2x \cdot 2 \cdot dx - \int \tan 2x \cdot dx$$

$$= \frac{1}{2} \cdot \frac{(\tan 2x)^2}{2} - \int \tan 2x \cdot dx$$

$$= \frac{1}{4} \cdot \tan^2 2x - \frac{1}{2}\int \tan 2x \cdot 2 \cdot dx$$

$$= \frac{1}{4} \cdot \tan^2 2x - \frac{1}{2} \cdot \ln|\cos 2x| + c$$

(Ans. $\mathbf{\frac{1}{4}\tan^2 2x + \frac{1}{2}\ln|\cos 2x| + c}$**)**

PROBLEM 5.46

Evaluate $\int \frac{\sec^2 x}{2 + \tan x} dx$

Solution:

$$\int \frac{\sec^2 x}{2 + \tan x} \cdot dx$$

$$\Rightarrow \quad \int \frac{\sec^2 x}{2 + \tan x} \cdot \sec^2 x \cdot dx$$

[Since $\int \frac{1}{u} du = \ln u + c$

let $\quad u = 2 + \tan x$

$du = \sec^2 x$]

$$\therefore \quad \int \frac{1}{2 + \tan x} \cdot \sec^2 dx = \ln(2 + \tan x) + c$$

(Ans. $\mathbf{\ln(2 + \tan x) + c}$**)**

PROBLEM 5.47

Evaluate $\int \sec^4 3x\,dx$

Solution: $\int \sec^4 3x\,dx$

$\Rightarrow \quad \int \sec^2 3x \cdot \sec^2 3x \cdot dx$

$\Rightarrow \quad \int \sec^2 3x \cdot (\tan^2 3x + 1) \cdot dx$

$\Rightarrow \quad \int \sec^2 3x \cdot (\tan^2 3x \cdot dx + \int \sec^2 3x \cdot dx$

$\because \quad \int \sec^2 u \cdot du = \tan u + c$

$$u = 3x$$
$$du = 3$$
$$f(x) = \tan 3x$$
$$f'(x) = \sec^2 3x \cdot 3$$

$\therefore$
$$\int \sec^4 3x \cdot dx = \frac{1}{3}\int \tan^2 3x \cdot \sec^2 3x \cdot 3 \cdot dx + \frac{1}{3}\int \sec^2 3x \cdot 3 \cdot dx$$
$$= \frac{1}{3} \cdot \frac{\tan^3 3x}{3} + \frac{1}{3} \cdot \tan 3x + c$$

(Ans. $\frac{1}{9}\tan^3 3x + \frac{1}{3}\tan 3x + c$)

PROBLEM 5.48

Evaluate $\int \frac{e^t}{1+e^{2t}}\,dt$

Solution: $\int \frac{e^t}{1+e^{2t}} \cdot dt$

$\Rightarrow \quad \int \frac{1}{1+e^{2t}} \cdot (e^t \cdot dt)$

$$= \int \frac{1}{1+(e^t)^2} \cdot (e^t \cdot dt)$$

$\because \quad \frac{1}{a^2+u^2}du = \tan^{-1} u + c$

$$u = e^t$$

$$du = e^t$$

$\because$ $$\int \frac{e^t}{1+e^{2t}} dt = \tan^{-1} e^t + c$$

(Ans. $\tan^{-1} e^t + c$)

PROBLEM 5.49

Evaluate $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$

Solution:

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}} \cdot dx = \int \cos\sqrt{x} \cdot \frac{dx}{\sqrt{x}}$$

$\because$ $$u = \sqrt{x}\, du = \frac{1}{2} x^{-\frac{1}{2}}$$

$$(x)^{\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$\therefore$ $$\int \frac{\cos\sqrt{x}}{\sqrt{x}} \cdot dx = 2\int \cos\sqrt{x} \cdot \frac{dx}{2\sqrt{x}}$$

$$= 2\sin\sqrt{x} + c$$

(Ans. $2\sin\sqrt{x} + c$)

PROBLEM 5.50

Evaluate $\int \frac{dx}{\sin x \,.\, \cos x}$

Solution: $$\int \frac{dx}{\sin x \cdot \cos x} = \int \frac{2dx}{2\sin x \cdot \cos x} = \int \frac{2dx}{\sin 2x} = \int 2\csc 2x\, dx$$

Since, $$\int \csc u\, du = -\ln|\csc u + \cot u| + c$$

$$\frac{dx}{\sin x \cos x} = \int \csc(2x) \cdot 2 \cdot dx$$

$$= -\ln|\csc 2x + \cot 2x| + c$$

(Ans. $-\ln|\csc 2x + \cot 2x| + c$)

PROBLEM 5.51

Evaluate $\int \sqrt{1+\sin y}\, dy$

Solution:

$$\int \sqrt{1+\sin y} \times \frac{\sqrt{1-\sin y}}{\sqrt{1-\sin y}} dy \Rightarrow \int \frac{\sqrt{1-\sin^2 y}}{\sqrt{1-\sin y}} dy$$

$$1-\sin^2 y = \cos^2 y = \int \sqrt{\cos^2 y} \cdot (1-\sin y)^{-\frac{1}{2}} \cdot dy\ f(x) = (1-\sin y)$$

$$f'(x) = -\cos y$$

$$\int \cos y.(1-\sin y)^{-\frac{1}{2}} \cdot dy = -\frac{(1-\sin y)^{\frac{1}{2}}}{\frac{1}{2}} + c = -2\sqrt{1+\sin y} + c$$

(Ans. $-2\sqrt{1-\sin yc}$**)**

PROBLEM 5.52

Evaluate $\int \frac{dx}{(x^2+1)(2+\tan^{-1}x)}$

Solution:

$$\int \frac{dx}{(x^2+1)(2+\tan^{-1}x)} \Rightarrow \int \frac{1}{(2+\tan^{-1}x)} \cdot \frac{dx}{(1+x^2)}$$

$\because$ $$\int \frac{1}{u} du = \ln u + c$$

$$u = 2+\tan^{-1}x$$

$$du = \frac{1}{1+x^2}$$

$$\frac{d}{du}\tan^{-1} u = \frac{1}{1+u^2} du$$

$$\int \frac{dx}{(x^2+1)(2+\tan^{-1}x)}$$

$$= \ln(2+\tan^{-1}x) + c$$

(Ans. $\ln(2+\tan^{-1}x)+c$**)**

PROBLEM 5.53

Evaluate $\int \sin^{-1}(\cosh x)\frac{\sinh x\, dx}{\sqrt{1-\cosh^2 x}}$

Solution:

$$\int \sin^{-1}(\cosh x)\frac{\sinh x \cdot dx}{\sqrt{1-\cosh^2 x}}$$

$$\because \qquad f(x) = \sin^{-1}(\cosh x)$$

$$f'(x) = \frac{1}{\sqrt{1-\cosh^2 x}} \cdot \sinh x$$

$$u = \cosh x$$

$$du = \sinh x$$

$$\therefore \qquad \int \sin^{-1}(\cosh x)\frac{\sinh x \cdot dx}{\sqrt{1-\cosh^2 x}} = \frac{(\sin^{-1}(\cosh x))^2}{2} + c$$

(Ans. $\frac{1}{2}(\sinh^{-1}(\cosh x))^2 + c$**)**

PROBLEM 5.54

Evaluate $\int \frac{\cos\theta d\theta}{1-\sin^2\theta}$

Solution:

$$\int \frac{\cos\theta d\theta}{1-\sin^2\theta}$$

$$\Rightarrow \qquad \int \frac{\cos\theta d\theta}{\cos^2\theta} \Rightarrow \int \frac{1}{\cos\theta} \cdot d\theta$$

$$= \int \sec\theta\, d\theta \Rightarrow \ln|\sec\theta + \tan\theta| + c$$

(Ans. $\ln|\sec\theta + \tan\theta| + c$**)**

PROBLEM 5.55

Evaluate $\int \frac{dx}{x(1+(\ln x)^2)}$

Solution:

$$\int \frac{dx}{x(1+(\ln x)^2)} = \int \frac{1}{(1+(\ln x)^2)} \frac{dx}{x}$$

$$= \tan^{-1}(\ln x) + c$$

Since

$$\int \frac{1}{a^2+u^2} du = \tan^{-1}\frac{u}{a} + c, \quad u = \ln x \, du = \frac{1}{x}$$

(Ans. $\tan^{-1}(\ln x) + c$)

PROBLEM 5.56

Evaluate $\int \left(e^{\frac{9}{4}x} - 2e^{\frac{5}{4}x} + e^{\frac{x}{4}} \right) dx$

Solution:

$$\int \left(e^{\frac{9}{4}x} - 2e^{\frac{5}{4}x} + e^{\frac{x}{4}} \right) dx$$

$$\Rightarrow \quad \int e^{\frac{9}{4}x} dx - \int 2e^{\frac{5}{4}x} dx + \int e^{\frac{x}{4}} dx$$

As

$$\int e^u du = e^u + c$$

$$\frac{4}{9}\int e^{\frac{9}{4}x} \cdot \left(\frac{9}{4}dx\right) - 2\frac{4}{5}\int e^{\frac{5}{4}} \cdot \left(\frac{5}{4}dx\right) + 4\int e^{\frac{x}{4}} dx$$

$$= \frac{4}{9}e^{\frac{9}{4}x} - \frac{8}{5}e^{\frac{5}{4}x} + 4e^{\frac{x}{4}} + c$$

(Ans. $\frac{4}{9}e^{\frac{9}{4}x} - \frac{8}{5}e^{\frac{5}{4}x} + 4e^{\frac{x}{4}} + c$)

PROBLEM 5.57

Evaluate $\int \frac{e^x dx}{e^{2x} + 2e^x + 1}$

Solution:

$$\int \frac{e^x dx}{e^{2x} + 2e^x + 1}$$

As $$e^{2x}+2^{ex}+1=(e^x+1)^2$$

$\therefore$ $$\int \frac{e^x}{(e^x+1)^2}dx \Rightarrow \int (e^x+1)^{-2}(e^x dx)$$

if $f(x)=e^x+1, f'(x)=e^x$

$\therefore$ $$\int \frac{e^x dx}{e^{2x}+2e^x+1}=\frac{(e^x+1)^{-1}}{-1}+c=-\frac{1}{(e^x+1)}$$

(Ans. $-\dfrac{1}{e^x+1}+c$**)**

PROBLEM 5.58

Evaluate $\int e^x \cdot \sinh 2x\, dx$

Solution: $$\int e^x \cdot \sinh 2x\; dx \text{ since } \sinh x=\frac{e^x-e^{-x}}{2}$$

$\therefore$ $$\int e^x\left(\frac{e^{2x}-e^{-2x}}{2}\right)dx \Rightarrow \int \frac{e^{3x}-e^{-x}}{2}dx$$

$$=\frac{1}{2}\int\left(e^{3x}-e^{-x}\right)dx \Rightarrow \frac{1}{2}\left(\int e^{3x}dx-\int e^{-x}dx\right)$$

$$=\frac{1}{2}\left(\frac{1}{3}e^{3x}3dx\right)-\left(-\int e^{-x}\cdot -1dx\right)$$

$$=\frac{1}{2}\frac{1}{3}e^{3x}+\frac{1}{2}e^{-x}+c$$

$$=\frac{1}{6}e^{3x}+\frac{1}{2}\frac{1}{e^x}+c$$

(Ans. $\dfrac{1}{2}\left[\dfrac{1}{3}e^{3x}+e^{-x}\right]+c$**)**

PROBLEM 5.59

Evaluate $\int \dfrac{\sec^3 x+e^{\sin x}}{\sec x}dx$

Solution:

$$\int \frac{\sec^3 x+e^{\sin x}}{\sec x}dx \Rightarrow \int (\sec^2 x+e^{\sin x})\cos x\, dx$$

Since, $\int \sec^2 u \cdot du = \tan u + c$

$$\frac{1}{\cos x} = \sec x$$

$\therefore$ $$\frac{1}{\cos x} = \cos x$$

$\therefore$ $$\int \frac{\sec^3 x + e^{\sin x}}{\sec x} dx = \tan x + e^{\sin x} + c$$

(Ans. $\tan x + e^{\sin x} + c$)

PROBLEM 5.60

Evaluate $\int \frac{3^{x+2}}{2+9^{x+1}} dx$

Solution:

$$\int \frac{3^{x+2}}{2+9^{x+1}} dx$$

$$3^{x+1} \cdot 3^1 = 3^{x+2}$$

$$9^{x+1} = (3^{x+1})^2$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$$

and $$a^2 = 2a = \sqrt{2}$$

$$u = 3^{x+1} du = 3^{x+1} \cdot \ln 3$$

$$\frac{d}{du} a^u = a^u \ln a \frac{du}{dx}$$

$\therefore$ $$3\int \frac{3^{x+1}}{2+(3^{x+1})^2} dx$$

$\Rightarrow$ $$\frac{3}{\ln 3}\int \frac{1}{2+(3^{x+1})^2} 3^{x+1} \cdot \ln 3\, dx$$

$$= \frac{3}{\sqrt{2}\ln 3}\tan^{-1}\frac{3^{x+1}}{\sqrt{2}} + c$$

(Ans. $\frac{3}{\sqrt{2}\ln 3}\tan^{-1}\frac{3^{x+1}}{\sqrt{2}} + c$)

PROBLEM 5.61

Evaluate $\displaystyle\int \frac{\cos x\, dx}{\sqrt{\sin x}\cdot\sqrt{1-\sin x}}$

Solution:

$$\int \frac{1}{\sqrt{1-\sin x}}\cdot\frac{\cos x}{\sin x}\,dx$$

$\Rightarrow$ $$\int \frac{1}{\sqrt{1-\sin x}}\cdot\frac{\cos x}{\sin x}$$

As $$\sin x = \left(\sqrt{\sin x}\right)^2$$

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\frac{u}{a}+c$$

$$y=\sqrt{\sin x}$$

$\Rightarrow$ $$dy = \frac{1}{2}(\sin x)^{-\frac{1}{2}}\cdot\cos x = \frac{\cos x}{2\sqrt{\sin x}}$$

$\Rightarrow$ $$2\int \frac{1}{1-(\sin x)^2}\cdot\frac{\cos x}{2\sqrt{\sin x}}\,dx = 2\sin^{-1}(\sqrt{\sin x})+c$$

(Ans. $\mathbf{2\sin^{-1}\sqrt{\sin x}+c}$**)**

PROBLEM 5.62

Evaluate $\displaystyle\int \tan^5 x\, dx$

Solution: $\Rightarrow \int \tan x\cdot\tan^4 x\,dx$

$$=\int \tan x(\sec x^2-1)^2\,dx$$

$$\tan^2 x = (\sec^2 x - 1)$$

$$=\int \tan x(\sec^4 x - 2\sec^4 x + 1)\,dx$$

$$=\int \tan x\cdot\sec^4 x\,dx - \int \tan x\cdot 2\sec^2 x\,dx + \int \tan x\,dx$$

$$=\int (\tan x\cdot\sec x)\sec^3 x\,dx - 2\int \tan x\cdot\sec^2 x + \int \tan x\,dx$$

$$=\int (\tan x \cdot \sec x)\sec^3 x\,dx - 2\int \sec(\tan x \cdot \sec x)\,dx + \int \tan x\,dx$$

$$=\frac{1}{4}+\sec^4 x - \sec^2 x - \ln|\cos x| + c$$

(Ans. $\frac{1}{4}\sec^4 x - sec^2 x - \ln|\cos x| + c$**)**

PROBLEM 5.63

Evaluate $\int e^{\ln \sin^{-1} x} \cdot \frac{dx}{\sqrt{1-x^2}}$

Solution:

$$\int e^{\ln \sin^{-1} x}\frac{dx}{\sqrt{1-x^2}} = \int \sin^{-1} x \cdot \frac{dx}{\sqrt{1-x^2}}$$

$$=\frac{(\sin^{-1} x)^2}{2} + c$$

(Ans. $\frac{1}{2}(\sin^{-1} x)^2 + c$**)**

PROBLEM 5.64

Evaluate $\int x \cdot e^{x^2-1}dx$

Solution:

$$\int xe^{x^2-1}dx = \int e^u \cdot du = e^u + c$$

and $u = x^2 - 1du = 2x$

$\therefore$ $\int xe^{x^2-1}dx = \frac{1}{2}\int 2x\,e^{x^2-1}dx = \frac{1}{2}e^{x^2-1} + c$

(Ans. $\frac{1}{2}e^{x^2-1} + c$**)**

PROBLEM 5.65

Evaluate $\int \cosh(\ln \cos x)\,dx$

Solution:

$$\cosh x = \frac{e^x - e^{-x}}{2}$$

$$\Rightarrow \qquad \int \frac{e^{\ln\cos x} + e^{-\ln\cos x}}{2}dx = \frac{1}{2}\int\left[e^{\ln\cos x} + e^{-\ln\cos x}\right]dx$$

$$= \frac{1}{2}\int (\cos x + \sec x)dx$$

$$= \frac{1}{2}[\sin x + \ln|\sec x + \tan x|] + c$$

(Ans. $\mathbf{\frac{1}{2}[\sin x + \ln|\sec x + \tan x|] + c}$**)**

PROBLEM 5.66

Evaluate $\int \frac{\cos x}{\sin^2 x}dx$

Solution:

$$\int \frac{\cos x}{\sin^2 x}dx$$

$$\Rightarrow \qquad \int \frac{\cos x}{\sin x}\cdot\frac{1}{\sin x}dx$$

$$= \int \cot u \cdot \csc x\, dx$$

$$= -\csc + c$$

(Ans. $\mathbf{-\csc x + c}$**)**

PROBLEM 5.67

Evaluate $\int \cosh^{-1}(\sin x)\frac{\cos x\, dx}{\sqrt{\sin^2 x - 1}}$

Solution:

$$\int \cosh^{-1}(\sin x)\frac{\cos x\, dx}{\sqrt{\sin^2 x - 1}}$$

If $\qquad f(x) = \cosh^{-1}(\sin x)$

Then, $\qquad f'(x) = \frac{1}{\sqrt{\sin^2 x - 1}}.\cos x$

$$\therefore \qquad \int \cos h^{-1}(\sin x)\cdot\frac{\cos x\, dx}{\sqrt{\sin^2 x - 1}} = \frac{\cosh^{-1}(\sin x)^2}{2} + c$$

(Ans. $\mathbf{\frac{1}{2}\left[\cosh^{-1}(\sin x)\right]^2 + c}$**)**

CHAPTER 6

Methods of Integration

PROBLEMS

PROBLEM 6.1

Evaluate $\int \frac{x^3}{x-1}dx$

Solution: $\int \frac{x^3}{x-1}dx$

$$\int\left(x^2+x+1+\frac{1}{x-1}\right)dx=\frac{x^3}{2}+\frac{x^2}{2}+x+\ln(x-1)+c$$

(Ans. $\frac{1}{3}x^3+\frac{1}{2}x^2+x+\ln(x-1)+c$**)**

PROBLEM 6.2

Evaluate $\int \frac{3x+2}{3x-1}dx$

Solution: $\int \frac{3x-2}{3x-1}dx$

$$=\int 1+\frac{3}{3x-1}dx$$

$$=x+\ln(3x-1)+c$$

(Ans. $x+\ln(3x-1)+c$**)**

PROBLEM 6.3

Evaluate $\int x^2 \cdot e^{-x} dx$

Solution: $\int x^2 e^{-x} dx$

As $$u\,dv = uv - \int v\,du \int x^2 e^{-x} dx$$

Here, $$u = x^2 \qquad dv = e^{-x}$$

$$du = 2x \qquad v = -e^{-x}$$

$$\therefore \quad \int x^2 e^{-x} dx = -x^2 e^{-x} - \int -e^{-x} 2x$$

Integration by part $$u = 2x \qquad dv = -e^{-x}$$

$$du = 2 \qquad v = -e^{-x}$$

$$\Rightarrow \quad \int x^2 e^{-x} dx = -x^2 e^{-x} - 2x e^{-x} - \int e^{-x} \cdot 2$$

$$\Rightarrow \quad \int x^2 e^{-x} dx = -x^2 e^{-x} - 2x e^{-x} + \int e^{-x} \cdot 2$$

$$\Rightarrow \quad x^2 e^{-x} - 2x e^{-x} - 2e^{-x} - 2\int e^{-x} \cdot dx = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$$

$$\therefore \quad \int x^2 e^{-x} dx = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + c$$

(Ans. $-e^{-x}(x^2 + 2x + 2) + c$**)**

PROBLEM 6.4

Evaluate $\int x \cdot \sin x^2 dx$

Solution: $\int x \cdot \sin x^2 dx$

$$\int \sin u\,du = -\cos u + c$$

$$u = x^2 \qquad du = 2x$$

$$\frac{1}{2}\int 2x \sin x^2 dx = -\frac{1}{2}\cos x^2 + c$$

(Ans. $-\frac{1}{2}\cos x^2 + c$**)**

PROBLEM 6.5

Evaluate $\int \sqrt{x^2 - 1}\, dx$

Solution: $\int\sqrt{x^2-1}\,dx$

$$x=\sec\theta \quad dx=\sec\theta\cdot\tan\theta\cdot d\theta$$

$$\therefore \quad \sqrt{x^2-1}dx=\int\sqrt{\sec^2\theta-1}(\sec\theta\tan\theta)d\theta$$

$$=\int\tan\theta\cdot\sec\theta\cdot\tan\theta\cdot d\theta=\int\sec\theta\tan^2\theta\,d\theta$$

$$\tan^2\theta=\sec^2\theta-1$$

$$\Rightarrow \quad \int\sec\theta(\sec^2\theta-1)d\theta=\underset{I_1}{\int\sec^3\theta d\theta}-\underset{I_2}{\int\sec\theta-d\theta}$$

$$\text{Consider } I_1=\int\sec^3 d\theta=\int\sec^2\theta\cdot\sec\theta\,d\theta$$

Integrating by parts, we obtain

$$I_1=\sec\theta\tan\theta-\int\sec\theta\tan^2\theta\,d\theta$$

$$=\sec\theta\tan\theta-\int\sec\theta(\sec^2\theta-1)d\theta$$

$$=\sec\theta\tan\theta-\int\sec^2\theta\,d\theta+\int\sec\theta\,d\theta$$

$$2I_1=\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|$$

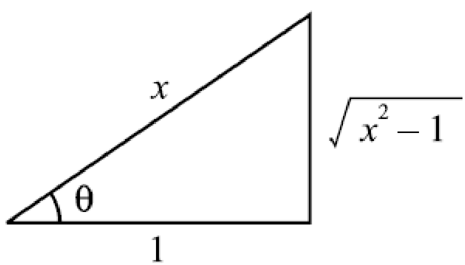
$$I_1=\frac{1}{2}\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|$$

$$\therefore \quad \int\sqrt{x^2}-dx=I_1-I_2$$

$$=\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta|-\ln|\sec\theta-\tan\theta|$$

$$=\frac{1}{2}\sec\theta\tan\theta-\frac{1}{2}\ln|\sec\theta+\tan\theta|$$

$$=\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|+c$$



PROBLEM 6.6

Evaluate $\displaystyle\int \frac{3x+13}{(5x-1)(7x+2)}dx$

Solution: $\displaystyle\int \frac{3x+13}{(5x-1)(7x+2)}dx$

Integrate by partial fraction

$$\frac{3x+13}{(5x-1)(7x+2)} = \frac{A}{(5x-1)} + \frac{B}{(7x+2)}$$

$$\therefore \quad \frac{3x+13}{(5x-1)(7x+2)} = \frac{A}{(5x-1)} + \frac{B}{(7x+2)} = \frac{(7x+2)A+(5x-1)B}{(5x-1)(7x+2)}$$

$$\therefore \quad 3x+13 = (7x+2)A+(5x-1)B$$

$$3x+13-7x\,A+2\,A+5xB-B$$

$$3x = 7xA+5xB \qquad \ldots(1)$$

$$13 = 2A-B \qquad \ldots(2)$$

$$3 = 7A+5B \qquad \ldots(1)$$

$$13 = 2A-B \qquad \ldots(2)$$

$$13 = 2A-B$$

$$13-B = 2A$$

$$3 = 7\left(\frac{13+B}{2}\right)+5B$$

$$3 = \frac{91+7B}{2}+5B$$

$$3 = \frac{91+7B+10B}{2}$$

$$\Rightarrow \quad 17b+91 = 6$$

$$17B = -85 \Rightarrow B = -5$$

$$A = \frac{13+B}{2} \quad \Rightarrow \quad \frac{13+(-5)}{2}$$

$$\therefore \quad A = 4$$

$$\int \frac{3x+13}{(5x-1)(7x+2)}dx = \int\left[\frac{4}{5x-1} - \frac{5}{7x+2}\right]dx$$

$$= \int \frac{4}{5x-1}dx - \int \frac{5}{7x+2}dx$$

$$= \frac{4}{5}\int \frac{5}{5x-1} - \frac{5}{7}\int \frac{7}{7x-2}dx$$

$$= \frac{4}{5}\ln(5x-1) - \frac{5}{7}\ln(7x+2) + c$$

(Ans. $\mathbf{\frac{4}{5}\ln(5x-1) - \frac{5}{7}\ln(7x+2) + c}$**)**

PROBLEM 6.7

Evaluate $\displaystyle\int \frac{2x-3}{(x-1)(x-2)(x+3)}dx$

Solution:

$$\int \frac{2x-3}{(x-1)(x-2)(x+3)}dx = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+3}$$

$\Rightarrow$ $$2x-3 = A(x-2)(x+3) + B(x-1)(x+3)C(x-1)(x-2)$$

at $$x = 1 \Rightarrow A = \frac{1}{4}, \quad \text{at } x = 2 \Rightarrow B = \frac{1}{5}, \quad \text{at } x = -3 \Rightarrow C = -\frac{9}{20}$$

$$\int \frac{2x-3}{(x-1)(x-2)(x+3)}dx = \int\left(\frac{\frac{1}{4}}{x-1} + \frac{\frac{1}{5}}{x-2} - \frac{\frac{9}{20}}{x+3}\right)dx$$

$$= \frac{1}{4}\ln|x-1| + \frac{1}{5}\ln|x-2| - \frac{9}{20}\ln|x+3| + c$$

(Ans. $\mathbf{\frac{1}{4}\ln|x-1| + \frac{1}{5}\ln|x-2| - \frac{9}{20}\ln|x+3| + c}$**)**

PROBLEM 6.8

Evaluate $\int \frac{dx}{x^4 - 1}$

Solution: $\int \frac{dx}{x^4 - 1}$

Integration by partial fraction

$$\Rightarrow \qquad \int \frac{dx}{x^4 - 1} = \int \frac{dx}{(x^2 - 1)(x^2 + 1)} \Rightarrow \int \frac{dx}{(x+1)(x-1)(x^2+1)}$$

$$\Rightarrow \qquad \int \frac{dx}{x^4 - 1} \Rightarrow \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx + D}{x^2 + 1}$$

$$\int \frac{dx}{(x+1)(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx + D}{x^2 + 1}$$

$$\int \frac{dx}{(x+1)(x-1)(x^2+1)} = \frac{A(x+1)(x^2+1) + B(x+1)(x^2+1) + Cx + D(x-1)(x+1)}{(x+1)(x-1)(x^2+1)}$$

$$\Rightarrow \qquad 1 = A(x+1)(x^2+1) + B(x+1)(x^2+1) + Cx + D(x-1)(x+1)$$

$$1 = Ax^3 + Ax + Ax^2 + A + Bx^2 + Bx + Bx^2 - B + Cx^3 - Cx + Dx^2 - D$$

$$A + B + C = 0 \qquad \ldots(1)$$

$$A - B + D = 0 \qquad \ldots(2)$$

$$A + B - C = 0 \qquad \ldots(3)$$

$$A - B - D = 1 \qquad \ldots(4)$$

From Equation (2) and Equation (4),

$$A - B + D = 0$$

$$-A + B + D = -1$$

$$2D = -1 \qquad \therefore \quad D = -\frac{1}{2}$$

From Equation (3), $A=-B$

From Equation (4), $A-B-D=1$

$$-B-B+\frac{1}{2}=1 \quad \Rightarrow \quad -2B=\frac{1}{2}$$

$\therefore$ $B=-\frac{1}{4}$

$\therefore$ $A=\frac{1}{4}$

From Equation (1) and Equation (3),

$$A+B+C=0$$

$$-A\pm B\pm C=0 \qquad \therefore 2C=0 \qquad \therefore C=0 \quad \textit{and} \quad D=-\frac{1}{2}$$

A, B, C, D

$$\int\frac{dx}{x^4-1}=\int\frac{A}{x-1}dx+\int\frac{B}{x+1}dx+\int\frac{Cx+D}{x^2+1}dx$$

$\Rightarrow$
$$\int\frac{dx}{x^4-1}=\int\frac{\frac{1}{4}}{x-1}dx+\int\frac{-\frac{1}{4}}{x+1}dx+\int\frac{-\frac{1}{2}}{x^2+1}dx$$

As $\int\frac{1}{u}\cdot du=\ln u+c$

$$u=(x+1)du=1$$

$$u=(x-1)du=1$$

$$\int\frac{1}{a^2+u^2}du=\frac{1}{a}\tan^{-1}\frac{u}{a}+c$$

$$u=x \qquad du=1$$

$$\int\frac{dx}{x^4-1}=\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{2}\tan^{-1}+c$$

$$=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}+c$$

(Ans. $\mathbf{\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+c}$**)**

PROBLEM 6.9

Evaluate $\int \ln x\, dx$

Solution: $\int \ln x\, dx$

Use by part integration:

$$\int u\, dv = uv - \int v\, du$$

Let

$$u = \ln x \qquad du = \frac{1}{x}\cdot dx$$

$$dv = dx \qquad v = x$$

$$\therefore \qquad \int \ln x\, dx = x \ln x - \int x \cdot \frac{1}{x} dx$$

$$= x \ln x - \int 1 dx \quad \Rightarrow \quad x \ln x - x + c$$

(Ans. $\boldsymbol{x \cdot \ln x - x + c}$**)**

PROBLEM 6.10

Evaluate $\int \tan^{-1} x\, dx$

Solution: $\int \tan^{-1} x dx \Rightarrow$ integration by parts

$$\int u\, dv = uv - \int v\, dv$$

Let

$$u = \tan^{-1} x \quad \Rightarrow \quad du = \frac{1}{1+x^2} dx$$

$$dv = dx \quad \Rightarrow \quad v = x$$

$$\int u\, dv = uv - \int v\, du$$

$$\int \tan^{-1} x\, dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$$

$$\int \frac{1}{u} \cdot du = \ln u + c$$

$$u = 1 + x^2$$

$$du = \ln x$$

$$\Rightarrow \qquad \int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\int \frac{x}{1+x^2}\cdot 2dx$$

$$\Rightarrow \qquad \int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c$$

(Ans. $\boldsymbol{x\cdot \tan^{-1}x - \frac{1}{2}\ln(1+x^2)+c}$**)**

PROBLEM 6.11

Evaluate $\boldsymbol{\int x\cdot \ln x\,dx}$

Solution: $\int x\cdot \ln x\,dx$

Integration by parts

Let $\quad u = \ln x \qquad\qquad du = \frac{1}{x}dx$

$\quad dv = x\,dx \qquad\qquad v = \frac{x^2}{2}$

$$\therefore \qquad \int u\,dv = uv - \int v\,du$$

$$\int x\ln x dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} - \frac{1}{x}dx$$

$$\int x\ln x dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int \frac{x^2}{x}dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\frac{x^2}{2} + c$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

(Ans. $\boldsymbol{\frac{x^2}{2}\ln x - \frac{x^2}{4} + c}$**)**

PROBLEM 6.12

Evaluate $\boldsymbol{\int x\tan^{-1}x\,dx}$

Solution: $\int x\cdot \tan^{-1} x\,dx$

Integration by parts

Let $u = \tan^{-1} x \qquad du = \frac{1}{1+x^2}dx$

$$dv = x\,dx \qquad v = \frac{x^2}{2}$$

$$\int u\,dv = uv - \int v\,du$$

$$\Rightarrow \qquad \int x\tan^{-1} x\,dx = \tan^{-1} x\frac{x^2}{2} - \int \frac{x^2}{2}\frac{1}{1+x^2}dx$$

$$= \tan^{-1} x\frac{x^2}{2} - \frac{1}{2}\int \frac{1}{1+x^2}$$

$$= \tan^{-1} x\frac{x}{2} - \frac{1}{2}\left[\int dx - \int \frac{1}{1+x^2}dx\right]$$

$$= \tan^{-1} x\frac{x^2}{2} - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + c$$

(Ans. $\mathbf{\frac{x^2}{2}\tan^{-1} x - \frac{1}{2}(x - \tan^{-1}x) + c}$**)**

PROBLEM 6.13

Evaluate $\int \mathbf{x^2 \cdot \cos ax\,dx}$

Solution: $\int x^2 \cos a\,x\,dx$

$$\int \cos u \cdot du = \sin u + c$$

$$\int \cos ax \cdot dx \Rightarrow \frac{1}{a}\int \cos ax = \frac{1}{a}\sin a + c$$

$$u = x^2 \qquad du = 2x$$

$$av = \cos ax \qquad dv = \frac{1}{a}\sin ax$$

Use integration by parts:

$$\int udv = uv - \int vdu$$

$$\therefore \qquad \int x^2 \cos ax = \frac{x^2}{a}\sin ax - \int \frac{1}{a}\sin ax \cdot 2x$$

Integration by parts

$$\int \frac{1}{a}\sin ax\, u = ax\, du\, a \frac{1}{a}\int \frac{1}{a}\sin ax\, dx \frac{1}{a^2}\int \sin ax\, dx$$

$$-\frac{1}{a^2}\cos ax + c\int -\frac{1}{a^2}\cos ax + c$$

$$u = ax \qquad dv = a$$

$$-\frac{1}{a^3}\int \cos ax \cdot x$$

$$\int x^2 \cos ax = \frac{x^2}{a}\sin ax + \frac{2x}{a^2}\cos ax - \frac{2}{a^3}\sin ax + c$$

(Ans. $\boldsymbol{\frac{x^2}{a}\sin ax - \frac{2x}{a^2}\cos ax - \frac{2}{a^3}\sin ax + c}$**)**

PROBLEM 6.14

Evaluate $\boldsymbol{\int \sin(\ln x)dx}$

Solution: $\int \sin(\ln x)dx$

Integration by parts

Let $$u = \sin(\ln x)$$

$$du = \cos(\ln x)\cdot \frac{1}{x}dx$$

$$dv = dx$$

$$v = x$$

$\therefore$ $$\int u\, dv = uv - \int v\, dv$$

$\therefore$ $$\int \sin(\ln x)dx = \sin(\ln x)x - \int x\cos(\ln x)dx$$

$$= \sin(\ln x)x - \int \cos(\ln x)dx$$

Integration by parts

Let $u = \cos(\ln x) \qquad dv = -\sin(\ln x)\,dx$

$dv = dx \qquad v = x$

$$\therefore \int \cos(\ln x)dx \cdot x\cos(\ln x) - \int -\frac{\sin(\ln x)}{x} x\,dx$$

$$= x\cos(\ln x) + \int \sin(\ln x)dx$$

$$\therefore \int \sin(\ln x)dx = \sin(\ln x)x - x\cos(\ln x) - \int \sin(\ln x)dx$$

$$2\int \sin(\ln x)dx = \sin(\ln x)x - x\cos(\ln x)$$

$$\therefore \int \sin(\ln x)dx = \frac{x}{2}\sin(\ln x) - \frac{x}{2}\cos(\ln x) + c$$

(Ans. $\frac{x}{2}\sin(\ln x) - \frac{x}{2}\cos(\ln x) + c$)

PROBLEM 6.15

Evaluate $\int \ln(a^2 + x^2)\,dx$

Solution: $\int \ln(a^2 + x^2)dx$

Integration by parts

$$u = \ln(a^2 + x^2)dx$$

$$du = \frac{2x}{a^2 + x^2}$$

$$\int \ln\left(a^2 + x^2\right)dx = x\ln(a^2 + x^2) - \int \frac{2x^2}{a^2 + x^2}dx$$

$$= x\ln(a^2 + x^2) - 2\int \frac{x^2}{a^2 + x^2}dx$$

$$\Rightarrow \int \ln(a^2 + x^2)dx = x\ln(a^2 + x^2) - 2\int\left(1 - \frac{a^2}{x^2 + a^2}\right)dx$$

$$\left[\text{As} \int \frac{du}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}\right.$$

$$\left. u = x \quad ; \quad du = 1\right]$$

$$= x\ln(a^2 + x^2) - 2x + \frac{a^2}{a}\tan^{-1}\frac{x}{a} + c$$

(Ans. $x\ln(a^2 + x^2) - 2x + \frac{a^2}{a}\tan^{-1}\frac{x}{a} + c$)

PROBLEM 6.16

Evaluate $\int x \cdot \sin^{-1} x\, dx$

Solution: $\int x \cdot \sin^{-1} x\, dx$

Use integration by parts:

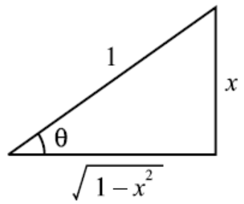
$$u = \sin^{-1} x \quad \Rightarrow \quad du = \frac{dx}{\sqrt{1-x^2}}$$

$$dv = x dx \quad \Rightarrow \quad v = \frac{x^2}{2}$$

$$\therefore \quad \int u\, dv = uv - \int v\, du$$

$$\Rightarrow \quad \int x \sin^{-1} x dx = \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{dx}{\sqrt{1-x^2}}$$

$$= \sin^{-1} x \cdot \frac{x^2}{2} - \frac{1}{2} \int \frac{x^2}{2} \cdot \frac{dx}{\sqrt{1-x^2}}$$



Integration by parts

$$\int \frac{x^2}{\sqrt{1-x^2}} dx$$

Let

$$x = \sin\theta \qquad dx = \cos d\theta$$

$$\int \frac{x^2}{\sqrt{1-x^2}} dx = \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos d\theta$$

$$1 - \sin^2\theta = \cos^2\theta$$

$$= \int \frac{\sin^2\theta}{\sqrt{1-\cos^2\theta}} \cos\theta\, d\theta$$

$$= \int \frac{\sin^2\theta}{\cos\theta}\cos\theta\, d\theta = \int \sin^2\theta\, d\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\Rightarrow \qquad \int \sin^2\theta\, d\theta = \int \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \int \frac{1}{2} d\theta - \frac{1}{2}\int \cos 2\theta\, d\theta$$

$$= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta = \frac{1}{2}(\theta - \sin\theta\cos\theta)$$

$$x = \sin\theta \qquad \therefore \quad \theta = \sin^{-1} x$$

$$\therefore \qquad \int x\sin^{-1} x\, dx = \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\left(\sin^{-1} x - x\sqrt{1-x^2}\right)$$

$$\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta$$

$$\sqrt{1-x^2} \Rightarrow \sqrt{1-\sin^2\theta} \Rightarrow \sqrt{\cos^2\theta} = \cos\theta$$

$$\therefore \qquad \int x\sin^{-1} x\, dx = \frac{x^2}{2}\sin^{-1} x - \frac{1}{4}\left(\sin^{-1} x - x\sqrt{1-x^2}\right) + c$$

(Ans. $\mathbf{\frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\sin^{-1}x + \frac{x}{4}\sqrt{1-x^2} + c}$**)**

PROBLEM 6.17

Evaluate $\int \cos^4 x\, dx$

Solution: $\int \cos^4 x\, dx \Rightarrow \int (\cos^2 x)^2\, dx$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\Rightarrow \qquad \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$\Rightarrow \qquad \int \frac{(1+\cos 2x)^2}{4} dx$$

$$\Rightarrow \quad \frac{1}{4}\int (1+\cos 2x)^2\,dx$$

$$\Rightarrow \quad \frac{1}{4}\int (1+2\cos 2x+\cos^2 2x)\,dx$$

$$=\frac{1}{4}\left[\int dx+\int 2\cos 2x\,dx+\int \cos^2 2x\,dx\right]$$

$$=\frac{1}{4}\left[x+\sin 2x+\int \cos^2 2x\,dx\right]$$

$$=\frac{1}{4}\left[x+\sin 2x+\frac{1}{2}\left(\frac{1+\cos 4x}{2}\right)\right]$$

$$=\frac{1}{4}\left[x+\sin 2x+\frac{1}{2}\left(\int dx+\frac{1}{4}\int \cos 4x\,4dx\right)\right]$$

$$=\frac{1}{4}\left[x+\sin 2x+\frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)\right]+c$$

$$=\frac{x}{4}+\frac{\sin 4x}{4}+\frac{x}{8}+\frac{1}{32}\sin 4x+c$$

$$=\frac{3}{8}x+\frac{\sin 2x}{4}+\frac{1}{32}\sin 4x+c$$

$$\left(\textbf{Ans. } \frac{3x}{8}+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+c\right)$$

PROBLEM 6.18

Evaluate $\int \cos^{\frac{2}{3}} x\cdot \sin^5 x\,dx$

Solution: $\int \cos^{\frac{2}{3}} x\cdot \sin^5 x\,dx$

$$\Rightarrow \quad \int \cos^{\frac{2}{3}} x\sin x(\sin^2 x)^2\,dx$$

$$\Rightarrow \quad \int \cos^{\frac{2}{3}} x\sin x(1-\cos^2 x)^2\,dx=\int \cos^{\frac{2}{3}} x\sin x(1-2\cos^2 x+\cos^4 x)dx$$

$$=\int\left[\cos^{\frac{2}{3}} x\sin x-2\cos^2 x\cdot\cos^{\frac{2}{3}}\sin x+\cos^{\frac{2}{3}} x\sin x\cos^4 x\right]dx$$

$$=\int\left[\cos^{\frac{2}{3}} x(-\sin x)-2\cos^{\frac{8}{3}} x(-\sin x)+\cos^{\frac{14}{3}} x(\sin x)\right]dx$$

$$= -\left(\frac{\cos^{\frac{5}{3}} x}{\frac{5}{3}} - 2\left(\frac{\cos^{\frac{11}{3}} x}{\frac{11}{3}} \right) + \frac{\cos^{\frac{17}{3}} x}{\frac{17}{3}} \right) + c$$

$$= -\frac{3}{5}\cos^{\frac{5}{3}} x + \frac{6}{11}\cos^{\frac{11}{3}} - \frac{3}{17}\cos^{\frac{17}{3}} x + c$$

(Ans. $\boldsymbol{-\frac{3}{5}\cos^{\frac{5}{3}} x + \frac{6}{11}\cos^{\frac{11}{3}} x - \frac{3}{17}\cos^{\frac{17}{3}} x + c}$**)**

PROBLEM 6.19

Evaluate $\int x \cdot \sin x \, dx$

Solution: $\int x \cdot \sin x \, dx$

Integration by parts $\int u\,dv = uv - \int v\,du$

Let $\quad u = x \qquad du = dx$

$\quad dv = \sin x\,dx \qquad v = -\cos x$

$$\int u\,dv = uv - \int v\,du$$

$$\int x \sin x\,dx = -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + c$$

(Ans. $\boldsymbol{-x \cdot \cos x + \sin x + c}$**)**

PROBLEM 6.20

Evaluate $\int x^2 \sqrt{1-x}\,dx$

Solution: $\int x^2 \sqrt{1-x}\,dx \Rightarrow \int x^2 (1-x)^{\frac{1}{2}}\,dx$

Use integration by parts:

$$\int u\,dv = uv - \int v\,du$$

Let $\quad u = x^2 \qquad du = 2x$

$$dv = (1-x)^{\frac{1}{2}}dx - \int (1-x)^{\frac{1}{2}}dx$$

$$\therefore \qquad v = -\frac{(1-x)^{\frac{3}{2}}}{\frac{3}{2}} = -\frac{2}{3}(1-x)^{\frac{3}{2}}$$

$$\therefore \qquad \int x^2(1-x)^{\frac{1}{2}}dx \Rightarrow \frac{-2x^2}{3}(1-x)^{\frac{3}{2}} - \int \frac{-2}{3}(1-x)^{\frac{3}{2}}2x\,dx$$

$$\therefore \qquad \int x^2(1-x)^{\frac{1}{2}}dx = -\frac{2}{3}x^2(1-x)^{\frac{3}{2}} - \frac{8x}{15}(1-x)^{\frac{5}{2}} - \frac{16}{105}(1-x)^{\frac{7}{2}} + c$$

$$= \frac{-2}{105}\sqrt{(1-x)^3}\,\begin{matrix}(35x^2+28(1-x)+8(1-x)^2)+c \\ (35x^2+28x-28x^2+8(1-2x+x^2))\end{matrix}$$

$$= \frac{-2}{105}\sqrt{(1-x)^3}(15x^2+12x+8)+c$$

$$\textbf{(Ans.}\ -\frac{\mathbf{2}}{\mathbf{105}}\sqrt{(1-x)^3}(15x^2+12x+8)+c\textbf{)}$$

PROBLEM 6.21

Evaluate $\int \sin^2 x \cdot \cos^2 x\,dx$

Solution: $\int \sin^2 x \cdot \cos^2 x\,dx$

$$\int \sin^2 x\cos^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\cdot\frac{1}{2}(1+\cos 2x)dx$$

$$= \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)dx$$

$$= \frac{1}{4}\int (1-\cos^2 2x)dx$$

$$= \frac{1}{4}\int\left(1-\left(\frac{1+\cos 4x}{2}\right)\right)dx$$

$$= \frac{1}{4}\left(\int dx - \frac{1}{2}\int (1+\cos 4x)dx\right)$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 4x}{2}$$

$$\frac{1}{4}\int \cos 4x\,dx = \sin 4x$$

$$=\frac{1}{4}\left(\int dx-\frac{1}{2}\left(\int dx+\frac{1}{4}\int \cos 4x\,dx\right)\right)$$
$$=\frac{1}{4}x-\frac{1}{8}x-\frac{1}{32}\sin 4x+c$$
$$=\frac{1}{8}x-\frac{1}{32}\sin 4x+c$$
$$=\frac{1}{8}\left(x-\frac{1}{8}\sin 4x\right)+c$$

(Ans. $\mathbf{\frac{1}{32}(4x-\sin 4x)+c}$**)**

PROBLEM 6.22

Evaluate $\int \sec^3 x\cdot\tan^2 x\,dx$

Solution:

$$\int \sec^3 x\cdot\tan^2 x\,dx \qquad \ldots (A)$$

As $\tan^2 x=\sec^2 x-1$

$$\Rightarrow \int \sec^3 x\cdot(\sec^2 x-1)dx$$
$$\Rightarrow \int (\sec^5 x-\sec^3 x)dx$$
$$\Rightarrow \int \sec^5 x\,dx-\int \sec^3 x\,dx$$
$$\Rightarrow \int \sec^2 x\sec^3 x\,dx-\int \sec x\sec^2 x\,dx$$

Integration by parts

*to find $\int \sec x\sec^2 x\,dx=\int \sec^3 x\,dx$

Let $u=\sec x \Rightarrow du=\sec x\cdot\tan x\,dx$

$dv=\sec^2 x\,dx \Rightarrow v=\tan x$

$$\therefore \quad \int \sec^3 x\,dx=\sec x\cdot\tan x-\int \sec x\cdot\tan^2 x\,dx$$

$$= \sec x \cdot \tan x - \int \sec x \cdot (\sec^2 x - 1) dx$$

$$= \sec x \cdot \tan x - \int \sec^3 x\, dx - \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x \cdot \tan x - \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x \cdot \tan x + \ln|\sec^2 x + \tan^2 x| + c$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + c$$

$$\therefore \quad \int \sec^3 x\, dx = \frac{1}{2}\sec x \cdot \tan x + \frac{1}{2}\ln|\sec x + \tan x| + c$$

* to find $\int \sec^3 x\, dx = \int \sec^2 x \sec^3 x\, dx$

Let $u = \sec^3 x \quad \Rightarrow \quad du = 3\sec^3 x \cdot \tan x\, dx$

$dv = \sec^2 x\, dx \quad \Rightarrow \quad v = \tan x$

$$\therefore \quad \int \sec^5 x\, dx = \tan x \cdot \sec^3 x - 3\int \sec^3 x \cdot \tan^2 x\, dx$$

$$\therefore \quad \int \sec^3 x \tan^2 x\, dx = \tan x \cdot \sec^3 x - 3\int \sec^3 x \cdot \tan^2 x\, dx - \frac{1}{2}\sec x \cdot \tan x - \frac{1}{2}\ln|\sec x + \tan x| + c$$

$$4\int \sec^3 x \tan^2 x\, dx = \tan x \cdot \sec^3 x - \frac{1}{2}\sec x \cdot \tan x - \frac{1}{2}\ln|\sec x + \tan x| + c$$

$$\therefore \quad \int \sec^3 x \tan^2 x\, dx = \frac{1}{4}\tan x \cdot \sec^3 x - \frac{1}{8}\sec x \cdot \tan x - \frac{1}{8}\ln|\sec x + \tan x| + c$$

(Ans. $\mathbf{\frac{1}{4}\sec^3 x \cdot \tan x - \frac{1}{8}\sec x \cdot \tan x - \frac{1}{8}\ln|\sec x + \tan x| + c}$**)**

PROBLEM 6.23

Evaluate $\int x(\cos^3 x^2 - \sin^3 x^2)\, dx$

Solution: $\int x(\cos^3 x^2 - \sin^3 x^2)\, dx$

$$\Rightarrow \int x(\cos x^2 \cdot \cos^2 x^2)dx - \int x(\sin x^2 \cdot \sin^2 x^2)dx \qquad \begin{bmatrix} \because \cos^2 x + \sin^2 x = 1 \\ \cos^2 x = 1 - \sin^2 x \\ \sin^2 x = 1 - \cos^2 x \end{bmatrix}$$

$$\Rightarrow \int x(\cos x^2(1-\sin^2 x^2)dx - \int x\sin x^2(1-\cos^2 x^2)dx + \int x\sin x^2 \cos^2 x^2 dx$$

$$\Rightarrow \frac{1}{2}\int 2x\cos x^2 dx - \frac{1}{2}\int \sin^2 x^2(2x\cos x^2)dx - \frac{1}{2}\int 2x\sin x^2 dx - \frac{1}{2}\int \cos^2 x^2(-2x\sin x^2)dx$$

$$\Rightarrow \frac{1}{2}\left[\sin x^2 - \frac{1}{3}\sin^3 x^2 + \cos x^2 - \frac{1}{3}\cos^3 x^2\right] + c$$

(Ans. $\mathbf{\frac{1}{2}\sin x^2 - \frac{1}{6}\sin^3 x^2 + \frac{1}{2}\cos x^2 - \frac{1}{6}\cos^3 x^2 + c}$**)**

PROBLEM 6.24

Evaluate $\displaystyle\int \frac{dx}{\sqrt{x}\sqrt{1-x}}$

Solution: $\displaystyle\int \frac{dx}{\sqrt{x}\sqrt{1-x}}$

$$\Rightarrow \int \frac{dx}{\sqrt{1-x}} \cdot \frac{1}{\sqrt{x}} \qquad \left(\because \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c \right. \qquad \boxed{\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c}$$

$$\text{If} \quad u = \sqrt{x} = (x)^{\frac{1}{2}}$$

$$\left. du = \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2\sqrt{x}} \right)$$

$$\int \frac{dx}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{\sqrt{x}} = 2\int \frac{dx}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} = 2\sin^{-1}\sqrt{x} + c$$

(Ans. $\mathbf{2\sin^{-1}\sqrt{x} + c}$**)**

PROBLEM 6.25

Evaluate $\displaystyle\int \frac{dx}{\sqrt{x}\cdot(1+\sqrt{x})}$

Solution:

$$\int \frac{dx}{\sqrt{x}\cdot(1+\sqrt{x})} = \int \frac{dx}{(1+\sqrt{x})}\cdot\frac{1}{\sqrt{x}}dx \qquad \left[\text{As } \int \frac{1}{u}du = \ln|u| + c\right.$$

$$\Rightarrow \qquad 2\int \frac{dx}{(1+\sqrt{x})}\cdot\frac{1}{2\sqrt{x}} = 2\ln|1+\sqrt{x}| \qquad \text{Here,} \quad u = 1+\sqrt{x}$$

$$\left. du = \frac{1}{2\sqrt{x}}\right]$$

$$\left(\textbf{Ans. } 2\ \ln(1+\sqrt{x}) + c\right)$$

PROBLEM 6.26

Evaluate $\displaystyle\int \frac{dx}{x\sqrt{2-3\ln^2\sqrt{x}}}$

Solution: $\displaystyle\int \frac{dx}{x\sqrt{2-3\ln^2\sqrt{x}}} = \int \frac{dx}{x\sqrt{2-(\sqrt{3}\ln x)^2}}$

$$\left[\text{Here,} \qquad u = \frac{\sqrt{3}}{2}\ln x\right.$$

$$\therefore \qquad \left. du = \frac{\sqrt{3}}{2}\cdot\frac{1}{x}\right]$$

$$\therefore \qquad \int \frac{dx}{x\sqrt{2-3\ln^2\sqrt{x}}} = \frac{2}{\sqrt{3}}\int \frac{\frac{\sqrt{3}}{2x}dx}{\sqrt{2-(\sqrt{3}\ln x)^2}} = \frac{2}{\sqrt{3}}\sin^{-1}\frac{\sqrt{3}}{2}\ln x + c$$

$$\left(\textbf{Ans. } \frac{2}{\sqrt{3}}\sin^{-1}\left(\frac{\sqrt{3}}{2}\ln x\right) + c\right)$$

PROBLEM 6.27

Evaluate $\displaystyle\int \frac{e^{2x}dx}{\sqrt[3]{1+e^x}}$

Solution:

$$\int \frac{e^{2x}dx}{\sqrt[3]{1+e^x}}$$

$$\Rightarrow \qquad \int e^{2x}(1+e^x)^{\frac{-1}{3}}\,dx$$

$$\Rightarrow \qquad \int e^x \cdot e^x(1+e^x)^{\frac{-1}{3}}\,dx$$

Let $\qquad e^x = y$

$$e^x\,dx = dy$$

$$\Rightarrow \qquad \int e^x e^x dx(1+e^x)^{\frac{-1}{3}} = \int \frac{y\,dy}{\sqrt[3]{1+y}}$$

Use integration by parts

$$\int \frac{e^{2x}\,dx}{\sqrt[3]{1+e^x}} \Rightarrow \int \frac{e^x e^x \cdot dx}{\sqrt[3]{1+e^x}} = \int \frac{ydy}{\sqrt[3]{1+y}}$$

$$u = y \qquad \Rightarrow \qquad du = dy$$

$$dv = (1+y)^{\frac{-1}{3}} \qquad \Rightarrow \qquad v = \frac{(1+y)^{\frac{2}{3}}}{\frac{2}{3}}$$

$$\int u\,dv = uv - \int v\,du$$

$$= \frac{3}{2}u(1+y)^{\frac{2}{3}} - \int \frac{3}{2}(1+y)^{\frac{2}{3}}dy$$

$$= \frac{3}{2}y(1+y)^{\frac{2}{3}} - \frac{3}{2}\frac{(1+y)^{\frac{5}{3}}}{\frac{5}{3}} + c$$

$$= \frac{3}{2}e^x\sqrt[3]{(1+e^x)^2} - \frac{9}{10}\sqrt[3]{(1+e^x)^5} + c$$

(Ans. $\boldsymbol{\frac{3}{2}e^x\sqrt[3]{(1+e^x)^2} - \frac{9}{10}\sqrt[3]{(1+e^x)^5} + c}$**)**

PROBLEM 6.28

Evaluate $\displaystyle\int \frac{dy}{y(2y^3+1)^2}$

Solution: $\displaystyle\int \frac{dy}{y(2y^3+1)^2}$ $\qquad \displaystyle\frac{1}{y}=\frac{y^2}{y^3}$

$$\int \frac{y^2dy}{y^3(2y^3+1)^2}$$

Let $2y^3=\tan^2\theta \quad \Rightarrow \quad 6y^2dy=2\tan\theta\cdot\sec^2\theta\,d\theta$

$$\therefore \quad y^2\,dy=\frac{2\tan\theta\cdot\sec^2\theta d\theta}{6}$$

$$=\frac{1}{3}\tan\theta\cdot\sec^2\theta\,d\theta$$

$$\Rightarrow \int \frac{\frac{1}{3}\tan\theta\cdot\sec^2\theta\,d\theta}{\frac{\tan^2\theta}{2}(\tan^2\theta+1)^2}=\frac{2}{3}\int\frac{\tan\theta\cdot\sec^2\theta\,d\theta}{\tan^2\theta(\tan^2\theta+1)^2}$$

$\boxed{2y^3=\tan^2\theta \quad \therefore y^3=\frac{\tan^2\theta}{2}}$

$$=\frac{2}{3}\int\frac{\tan\theta\cdot\sec^2\theta\,d\theta}{\tan^2\theta\cdot(\sec^2\theta)^2}$$

$\boxed{\frac{1}{\tan\theta}=\cot\theta \qquad \sec^2\theta=\frac{1}{\cos^2\theta}}$

$$=\frac{2}{3}\int\frac{d\theta}{\tan^2\theta\cdot\sec^2\theta}$$

$$=\frac{2}{3}\int\cot\theta\cdot\cos^2\theta\,d\theta$$

$$=\frac{2}{3}\int\frac{\cos\theta}{\sin\theta}\cdot(1-\sin^2\theta)d\theta$$

$$=\frac{2}{3}\int\left(\frac{\cos\theta}{\sin\theta}-\frac{\cos\theta\sin^2\theta}{\sin\theta}\right)d\theta$$

$\boxed{\cos^2\theta=1-\sin^2\theta}$

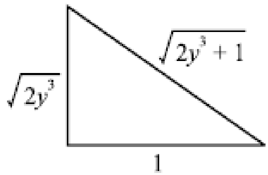
$$=\frac{2}{3}\int\left(\frac{\cos\theta}{\sin\theta}-\cos\theta\sin\theta\right)d\theta$$

$\boxed{\int\frac{1}{u}du=\ln|u|+c}$

$$=\frac{2}{3}\int\frac{\cos\theta}{\sin\theta}d\theta-\int\cos\theta\sin\theta d\theta$$

$$= \frac{2}{3}\left[\ln \sin\theta - \frac{\sin^2\theta}{2}\right] + c$$

$$= \frac{2}{3}\left[\ln\sqrt{\frac{2y^3}{2y^3+1}} - \frac{1}{2} + \frac{2y^3}{2y^3+1}\right] + c$$



(Ans. $\mathbf{\frac{1}{3}\ln\left(\frac{2y^3}{2y^3+1}\right) - \frac{2y^3}{3(2y^3+1)} + c}$**)**

PROBLEM 6.29

Evaluate $\displaystyle\int \frac{x\,dx}{1+\sqrt{x}}$

Solution: $\displaystyle\int \frac{x\,dx}{1+\sqrt{x}}$

Let $y = \sqrt{x}$

$\therefore$ $x = y^2$

$$dy = \frac{1}{2\sqrt{x}}dx \quad \Rightarrow \quad dy = \frac{1}{2y}dx$$

$\therefore$ $dx = 2ydy$

$\therefore$ $$\int \frac{x\,dx}{1+\sqrt{x}} \Rightarrow \int \frac{2y^3\,dy}{1+y}$$

$\Rightarrow$ $$\int \frac{y^3\,dy}{1+y} = 2\int\left(y^2 - y + 1 - \frac{1}{1+y}\right)dy$$

$$= 2\left[\frac{y^3}{3} - \frac{y^2}{2} + y - \ln(1+y)\right] + c$$

$$= \frac{2}{3}\sqrt{x^3} - x + 2\sqrt{x} - 2\ln(1-y) + c \qquad (\text{as}\, y = \sqrt{x})$$

(Ans. $\mathbf{\frac{2}{3}\sqrt{x^3} - x + 2\sqrt{2} - 2\ln(\sqrt{x}+1) + c}$**)**

PROBLEM 6.30

Evaluate $\int \frac{dt}{e^t - 1}$

Solution: $\int \frac{dt}{e^t - 1} \cdot \frac{e^t}{e^t}$

Let $e^t = x$

$$e^t dt = dx$$

$\because$ $x = e^t$

$$x dt = dx$$

$\therefore$ $dt = \frac{1}{x} dx$

$\int \frac{dx}{x(x-1)} \Rightarrow$ Integration by partial fraction

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1) + Bx}{x(x-1)}$$

$$1 = Ax - A + Bx$$

$$0 = Ax + Bx$$

$\therefore$ $A + B = 0$ ...(1)

$-A = 1$ ...(2)

$$A + B = 0$$

$\therefore$ $B = 1$

$$\boxed{\int \frac{1}{u} du = \ln u + c}$$

$$\int \frac{dx}{x(x-1)} = \int \frac{-1}{x} dx + \int \frac{1}{x-1} dx$$

$$dt = \frac{1}{x} dx$$

$\therefore$ $t = \ln x$

$$= -\ln x + \ln(x-1) + c$$

$$= -\ln e^t + \ln(e^t - 1) + c$$

$\therefore \Rightarrow$ $\ln(e^t - 1) - t + c$

(Ans. $\ln(e^t - 1) - t + c$)

PROBLEM 6.31

Evaluate $\int \frac{d\theta}{1-\tan^2\theta}$

Solution:

$$\int \frac{d\theta}{1-\tan^2\theta} = \int \frac{d\theta}{1-\frac{\sin^2\theta}{\cos^2\theta}} = \int \frac{d\theta}{\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}$$

$$= \int \frac{\cos^2\theta\, d\theta}{\cos^2\theta-\sin^2\theta} \qquad \left[\begin{array}{l} \cos^2\theta = \frac{1+\cos 2\theta}{2} \\ \sin^2\theta = \frac{1-\sin 2\theta}{2} \end{array}\right]$$

$$= \int \frac{\frac{1+\cos 2\theta}{2} d\theta}{\frac{1+\cos 2\theta}{2} - \frac{1-\cos 2\theta}{2}}$$

$$= \int \frac{\frac{1+\cos 2\theta}{2} d\theta}{\frac{(1+\cos 2\theta-(1-\cos^2\theta)}{2}} \qquad \Rightarrow \frac{\frac{1+\cos 2\theta}{2} 2d\theta}{1+\cos 2\theta - 1 + \cos 2\theta}$$

$$= \int \frac{1+\cos 2\theta\, d\theta}{2\cos 2\theta}$$

$$= \frac{1}{2}\int \left[\frac{1}{\cos 2\theta} + \frac{\cos 2\theta}{\cos 2\theta}\right] \cdot d\theta$$

As

$$\frac{1}{\cos 2\theta} = \sec^2\theta \qquad \text{and} \qquad \int \sec u \cdot du = \ln|\sec u + \tan u| + c$$

$$u = 2\theta \cdot du = 2$$

$$\therefore \quad \int \frac{d\theta}{1-\tan^2\theta} = \frac{1}{2}\left[\int \sec 2\theta \cdot do + \int d\theta\right] = \frac{1}{2}\left[\frac{1}{2}\int \sec 2\theta \cdot 2do + \int d\theta\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\ln|\sec 2\theta + \tan 2\theta| + \theta\right] = \frac{1}{4}\ln|\sec 2\theta + \tan 2\theta| + \frac{\theta}{2} + c$$

(Ans. $\mathbf{\frac{1}{2}\theta + \frac{1}{4}\ln|\sec 2\theta + \tan 2\theta| + c}$**)**

PROBLEM 6.32

Evaluate $\int \mathbf{e^x \cdot \cos 2x\, dx}$

Solution: $\int e^x \cdot \cos 2x\, dx$

Integration by parts

Let

$$u = \cos 2x \quad \Rightarrow \quad du = -2\sin 2x$$

$$dv = e^x dx \quad \Rightarrow \quad v = e^x$$

$$\int e^x \cos 2x\, dx = e^x \cos 2x + 2\int e^x \sin 2x\, dx$$

Integration by parts

Let

$$u = \sin 2x \quad \Rightarrow \quad du = 2\cos 2x\, dx$$

$$dv = e^x dx \quad \Rightarrow \quad v = e^x$$

$$\int e^x \cos 2x = e^x \cos 2x + 2e^x \sin 2x - 4\int e^x \cos 2x\, dx$$

$$5\int e^x \cos 2x = e^x \cos 2x + 2e^x \sin 2x$$

$$\therefore \quad \int e^x \cos 2x = \frac{1}{5}e^x \cos 2x + \frac{2}{5}e^x \sin 2x + c$$

(Ans. $\mathbf{\frac{e^x}{5}\cos 2x + \frac{2}{5}e^x \sin 2x + c}$**)**

PROBLEM 6.33

Evaluate $\int \frac{\cot\theta\, d\theta}{1+\sin^2\theta}$

Solution:

$$\int \frac{\cot\theta\, d\theta}{1+\sin^2\theta} = \int \frac{\cos\theta}{\sin\theta(1+\sin^2\theta)} d\theta$$

Let $x = \sin\theta$

$\therefore$ $dx = \cos\theta\, d\theta$

$\therefore$ $$\frac{\cos\theta\, d\theta}{\sin\theta(1+\sin^2\theta)} = \int \frac{dx}{x(1+x^2)}$$

Integration by partial fraction

$$\int \frac{dx}{x(1+x^2)} = \frac{1}{x(1+x^2)} = \frac{A}{x} = \frac{Bx+c}{1+x^2}$$

$$\frac{1}{x(1+x^2)} = \frac{A(1+x^2)+(Bx+c)(x)}{x(1+x^2)}$$

$$1 = A(1+x^2)+(Bx+c)(x)$$

$$1 = A + Ax^2 + Bx^2 + cx$$

$$Ax^2 + Bx^2 = 0$$

$$A + B = 0 \qquad \ldots(1)$$

$$cx = 0 \quad \Rightarrow \quad c = 0 \qquad \ldots(2)$$

$$A = 1 \qquad \ldots(3)$$

$$A + B = 0$$

$$1 + B = 0$$

$\therefore$ $B = -1$

$$c = 0$$

$\Rightarrow$ $$\int\left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx = \ln x - \frac{1}{2}\ln(1+x^2) + c$$

(Ans. $\ln\frac{\sin\theta}{\sqrt{1+\sin^2\theta}} + c$**)**

PROBLEM 6.34

Evaluate $\displaystyle\int \frac{e^{4t}}{(1+e^{2t})^{\frac{2}{3}}}dt$

Solution: $\displaystyle\int \frac{e^{4t}}{(1+e^{2t})^{\frac{2}{3}}}dt \Rightarrow \int \frac{e^{2t}\cdot e^{2t}dt}{(1+e^{2t})^{\frac{2}{3}}}$

Let $\quad x = e^{2t} \quad \Rightarrow \quad dx = 2e^{2t}\,dt$

$\therefore \quad dt = \dfrac{dx}{2e^{2t}} = \dfrac{dx}{2x}$

$\therefore \quad x = e^{2t} \quad \Rightarrow \quad x^2 = (e^{2t})^2 = e^{4t}$

$$\therefore \quad \int \frac{e^{4t}}{(1+e^{2t})^{\frac{2}{3}}}dt \Rightarrow \int \frac{x^2\cdot \frac{1}{2x}dx}{(1+x)^{\frac{2}{3}}}$$

$$\int \frac{\frac{x}{2}dx}{(1+x)^{\frac{2}{3}}} \Rightarrow \frac{1}{2}\int \frac{xdx}{(1+x)^{\frac{2}{3}}} = \frac{1}{2}\int x(1+x)^{-\frac{2}{3}}dx$$

$$\int x(1+x)^{-\frac{2}{3}}dx = \text{Integration by parts}$$

Let $\quad u = x \quad \Rightarrow \quad du = dx$

$$dv = (1+x)^{\frac{2}{3}}dx \quad \Rightarrow \quad v = \frac{(1+x)^{\frac{1}{3}}}{\frac{1}{3}} = 3(1+x)^{\frac{1}{3}}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int x(1+x)^{-\frac{2}{3}}dx = 3x(1+x)^{\frac{1}{3}} - \int 3(1+x)^{\frac{1}{3}}dx$$

$$= 3\times(1+x)^{\frac{1}{3}} - 3\int (1+x)^{\frac{1}{3}}dx$$

$$= 3\times(1+x)^{\frac{1}{3}} - 3\frac{(1+x)^{\frac{4}{3}}}{\frac{4}{3}} + c$$

$$= 3\times(1+x)^{\frac{1}{3}} - \frac{9}{4}(1+x)^{\frac{4}{3}} + c$$

$$\int \frac{e^{4t}dt}{(1+e^{4t})^{\frac{1}{3}}} = \frac{3e^{2t}}{2}(1+e^{2t})^{\frac{1}{3}} - \frac{9}{8}(1+e^{2t})^{\frac{4}{3}} + c$$

(Ans. $\mathbf{\frac{3}{2}e^{2t}(1+e^{2t})^{\frac{1}{3}} - \frac{9}{8}(1+e^{2t})^{\frac{3}{4}} + c}$**)**

PROBLEM 6.35

Evaluate $\displaystyle\int \frac{x^3+x^2}{x^2+x-2}dx$

Solution: $\displaystyle\int \frac{x^3+x^2}{x^2+x-2}dx$

$$\Rightarrow \qquad \int\left(x+\left(\frac{2x}{x^2+x-2}\right)\right)dx$$

As $\qquad x^2+x-2 = (x+2)\cdot(x-1)$

$$\Rightarrow \qquad \int\left(x+\frac{2x}{(x+2)\cdot(x-1)}dx\right)$$

Integration by partial fraction

$$\frac{2x}{(x+2)+(x-1)} = \frac{A}{x+2}+\frac{B}{x-1} \quad \Rightarrow \quad \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

$\therefore \qquad 2x = Ax - A + Bx + 2$

$2 = A + B \qquad \ldots(1)$

$\Rightarrow \qquad A = 2 - B$ should be substituted into Equation (2)

$0 = 2B - A \qquad \ldots(2)$

$\Rightarrow \qquad 0 = 2B-(2-B)$

$2B = (2-B)$

$\therefore \qquad 2B + B = 2$

$\therefore \qquad B = \frac{2}{3}$

$\therefore \qquad A = 2 - \frac{2}{3} \quad \Rightarrow \quad \frac{6-2}{3} = \frac{4}{3}$

$$\therefore \qquad \int \frac{x^3+x^2}{x^2+x^{-2}}dx = \int\left(x+\frac{\frac{4}{3}}{x+2}+\frac{\frac{2}{3}}{x-1}\right)dx$$

$$=\frac{x^2}{2}+\frac{4}{3}\ln(x+2)+\frac{2}{3}\ln(x-1)+c$$

(Ans. $\boldsymbol{\frac{x^2}{2}+\frac{4}{3}\ln(x+2)+\frac{2}{3}\ln(x-1)+c}$**)**

PROBLEM 6.36

Evaluate $\boldsymbol{\int \frac{2e^{2x}-e^x}{\sqrt{3e^{2x}-6e^x-1}}dx}$

Solution: $\int \frac{2e^{2x}-e^x}{\sqrt{3e^{2x}-6e^x-1}}dx$

$$\Rightarrow \qquad \frac{1}{\sqrt{3}}\int \frac{(2e^x-1)e^x dx}{\sqrt{e^{2x}-2e^x-\frac{1}{3}}} = \frac{1}{\sqrt{3}}\int \frac{(2e^x-1)e^x dx}{\sqrt{e^{2x}-2e^x-1-\frac{4}{3}}}$$

$$1-\frac{4}{3}=\frac{3}{4}-\frac{4}{3}=-\frac{1}{3}$$

$$e^{2x}-2e^x+1=(e^x-1)^2$$

$$=\frac{1}{\sqrt{3}}\int \frac{(2e^x-1)e^x\,dx}{\sqrt{(e^x-1)^2-\frac{4}{3}}}$$

Let $\qquad e^x-1=\frac{2}{\sqrt{3}}\sec\theta \quad \Rightarrow \quad e^x=\left(\frac{2}{\sqrt{3}}\sec\theta+1\right)$

$$e^x\,dx=\frac{2}{\sqrt{3}}\sec\theta\cdot\tan\theta\cdot d\theta$$

$$\therefore \quad \frac{1}{\sqrt{3}}\int \frac{2\left(\frac{2}{\sqrt{3}}\sec\theta+1\right)-1}{\sqrt{\frac{4}{3}\sec^2\theta-\frac{4}{3}}}\cdot\frac{2}{\sqrt{3}}\sec\theta\cdot\tan\theta\cdot d\theta$$

$$\therefore \quad \frac{1}{\sqrt{3}}\int \frac{\frac{4}{\sqrt{3}}\sec\theta + 2 - 1}{\sqrt{\frac{4}{3}\left(\sec^2\theta - 1\right)}} \cdot \frac{2}{\sqrt{3}}\sec\theta \cdot \tan\theta \cdot d\theta$$

$$(\sec^2\theta - 1) = \tan^2\theta$$

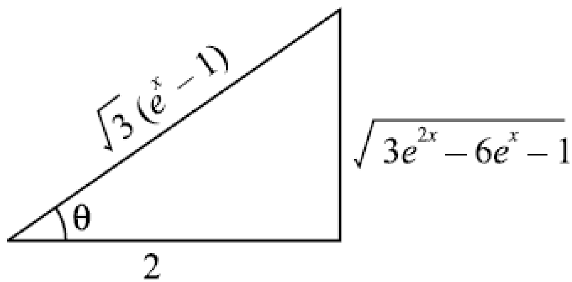
$$= \frac{1}{\sqrt{3}}\int \frac{\left(\frac{4}{\sqrt{3}}\sec\theta + 1\right)}{\frac{2}{\sqrt{3}}\sqrt{\tan^2\theta}} \cdot \frac{2}{\sqrt{3}}\sec\theta \cdot \tan\theta \cdot d\theta$$

$$= \frac{1}{\sqrt{3}}\int \frac{\left(\frac{4}{\sqrt{3}}\sec\theta + 1\right)}{\frac{2}{\sqrt{3}}\tan\theta} \cdot \frac{2}{\sqrt{3}}\sec\theta \cdot \tan\theta \cdot d\theta$$

$$= \frac{1}{\sqrt{3}}\int \left(\frac{4}{\sqrt{3}}\sec\theta + 1\right) \cdot \sec\theta \cdot d\theta$$

$$= \frac{4}{3}\tan\theta + \frac{1}{\sqrt{3}}\ln|\sec\theta + \tan\theta| + c$$

$$= \frac{4}{3}\frac{\sqrt{3e^{2x} - 6e^x - 1}}{2} + \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}\left(e^x - 1\right)}{2} + \frac{\sqrt{3e^{2x} - 6e^x - 1}}{2}\right| + c$$



(Ans. $\mathbf{\frac{1}{3}\left(2\sqrt{3e^{2x} - 6e^x - 1} + \sqrt{3}\ln\left|\sqrt{3}\,(e^x - 1) + \sqrt{3e^{2x} - 6e^x - 1}\right| + c\right)}$**)**

PROBLEM 6.37

Evaluate $\int \frac{dy}{(2y+1)\sqrt{y^2+y}}$

Solution: $\int \frac{dy}{(2y+1)\sqrt{y^2+y}}$

As $\int \frac{du}{u\sqrt{u^2+a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right|$

and if $u = 2y+1, \quad du = 2$

$\sqrt{4} = 2$

Also $u = 2y+1$

$u^2 = (2y+1)^2 \quad \Rightarrow \quad 4y^2+4y+1$

$$\therefore \quad \int \frac{2dy}{(2y+1)\cdot 2\sqrt{y^2+y}} = \int \frac{2dy}{(2y+1)\cdot\sqrt{4}\cdot\sqrt{y^2+y}} = \int \frac{2dy}{(2y+1)\cdot\sqrt{4y^2+4y}}$$

$$\Rightarrow \quad \int \frac{2dy}{(2y+1)\cdot\sqrt{(2y+1)^2}} = \sec^{-1}(2y+1)+c$$

(Ans. $\sec^{-1}(2y+1)+c$)

PROBLEM 6.38

Evaluate $\int (1-x^2)^{\frac{3}{2}}dx$

Solution: $\int (1-x^2)^{\frac{3}{2}}dx$

Let $x = \sin\theta \quad \Rightarrow \quad dx = \cos\theta\, d\theta$

$$\int (1-x^2)^{\frac{3}{2}}dx \Rightarrow \int (1-\sin^2\theta)^{\frac{3}{2}}\cos\theta d\theta$$

$$= \int (\cos^2\theta)^{\frac{3}{2}}\cos\theta\, d\theta$$

$$= \int \cos^{\frac{6}{2}}\theta\cdot\cos\theta\, d\theta = \int \cos^3\theta\cdot\cos\theta\, d\theta$$

$$= \int \cos^4 \theta \, d\theta = \int (\cos^2 \theta)^2 \, d\theta \qquad \left[\cos^2 \theta = 1 + \frac{1+\cos 2\theta}{2} \right]$$

$$= \int \left(\frac{1+\cos 2\theta}{2} \right)^2 d\theta$$

$$= \frac{1}{4} \int (1+\cos 2\theta)^2 \, d\theta \Rightarrow \frac{1}{4} \int (1 + 2\cos 2\theta + \cos^2 2\theta) d\theta$$

$$\Rightarrow \frac{1}{4} \left[\int d\theta + \int 2\cos 2\theta \, d\theta + \int \cos^2 2\theta \, d\theta \right]$$

$$\cos^2 2\theta = 1 + \frac{\cos 4\theta}{2}$$

$$\Rightarrow \frac{1}{4} \left[\int d\theta + \int 2\cos 2\theta \, d\theta + \int \left(\frac{1+\cos 4\theta}{2} \right) d\theta \right]$$

$$= \frac{1}{4} \left[\theta + \sin 2\theta + \frac{1}{2} \left(\theta + \frac{1}{4} \sin 4\theta \right) \right] = \frac{1}{4} \left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right] + c$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\sin 4\theta = 2 \sin 2\theta \cos 2\theta$$

$$\int (1+x^2)^{3/2} dx = \frac{3}{8} \sin^{-1} x + \frac{8}{8} \sqrt{1+x^2} (5 - 2x^2) + c$$

(Ans. $\boldsymbol{\frac{e^x}{5} \cos 2x + \frac{2}{5} e^x \sin 2x + c}$**)**

PROBLEM 6.39

Evaluate $\displaystyle\int \frac{\tan^{-1} x}{x^2} dx$

Solution: $\displaystyle\int \frac{\tan^{-1} x}{x^2} dx$

Let

$$u = \tan^{-1} x \quad \Rightarrow \quad du = \frac{1}{1+x^2} dx$$

$$dv = \frac{1}{x^2} dx \quad \Rightarrow \quad v = -\frac{1}{x^2}$$

Use integration by parts:

$$\int u\,dv = uv - \int v\,du$$

$$\int \frac{\tan^{-1} x}{x^2}\,dx = -\frac{1}{x^2}\tan^{-1} x + \int \frac{dx}{x(1+x^2)}$$

To find $\int \frac{dx}{x(1+x^2)} \Rightarrow$ use integration by partial fraction.

$$\int \frac{dx}{x(1+x^2)} = \int \frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+c}{1+x^2}$$

$$\frac{1}{x(1+x^2)} = \frac{A(1+x^3)+(Bx+c)(x)}{x(1+x^2)}$$

$$1 = A + Ax^2 + Bx^2 + cx$$

$$1 = A \qquad \ldots(1)$$

$$A = 1$$

$$0 = A + B \qquad \ldots(2)$$

$$\Rightarrow \quad 0 = 1 + B$$

$$\therefore \quad B = -1$$

$$0 = c \qquad \ldots(3)$$

$$\therefore \quad c = 0$$

$$\therefore \quad \int \frac{dx}{x(1+x^2)} = \int \frac{1}{x}\,dx + \int \frac{-x}{1+x^2}\,dx$$

$$\therefore \quad \int \frac{\tan^{-1} x}{x^2}\,dx = -\frac{1}{x}\tan^{-1} x + \int \frac{1}{x}\,dx + \int \frac{-x}{1+x^2}\,dx = -\frac{1}{x}\tan^{-1} x + \ln x + \frac{1}{2}\int \frac{-2x}{1+x^2}\,dx$$

$$= -\frac{1}{x}\tan^{-1} x + \ln x - \frac{1}{2}\ln(1+x^2) + c$$

$$= -\frac{1}{x}\tan^{-1}x + \ln \frac{x}{(1+x^2)^{\frac{1}{2}}} + c$$

$$= -\frac{1}{x}\tan^{-1}x + \ln \frac{x}{\sqrt{1+x^2}} + c$$

(Ans. $\mathbf{\ln \frac{x}{\sqrt{x^2+1}} - \frac{\tan^{-1} x}{x} + c}$**)**

PROBLEM 6.40

Evaluate $\int x \cdot \sin^2 x\, dx$

Solution: $\int x \cdot \sin^2 x\, dx$

$$\int x \frac{(1-\cos 2x)}{2} dx = \frac{1}{2}\int (x - x\cos 2x) dx$$

$$u = x \qquad du = dx$$

$$dv = \cos 2x\, dx \quad \Rightarrow \quad v = \frac{1}{2}\sin 2x$$

$$\int x \sin^2 x\, dx = \frac{1}{2}\left[\frac{x^2}{2} - \frac{x}{2}\sin^2 x - \frac{1}{4}\cos 2x\right] + c$$

$$\int \sin u \cdot du = -\cos u + c$$

$$= \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{\cos 2x}{8} + c$$

(**Ans.** $\frac{x^2}{4} - \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x + c$)

PROBLEM 6.41

Evaluate $\int \frac{dt}{t^4 + 4t^2 + 3}$

Solution: $\int \frac{dt}{t^4 + 4t^2 + 3}$

$$t^4 + 4t^2 + 3 \Rightarrow (t^2+3)(t^2+1)$$

$\int \frac{dt}{(t^2+3)(t^2+1)}$ use integration by partial fraction

$$\frac{1}{(t^2+3)(t^2+1)} = \frac{At+B}{(t^2+3)} + \frac{Ct+D}{(t^2+1)}$$

$$\frac{1}{(t^2+3)(t^2+1)} = \frac{(At+\beta)(t^2+1) + (Ct+D)(t^2+3)}{(t^2+3)(t^2+1)}$$

$$1 = At^3 + Bt^2 + At + B + Ct^3 + Dt^2 + 3Ct +$$

$$A + C = 0 \qquad \ldots(1)$$
$$B + D = 0 \qquad \ldots(2)$$
$$A + 3C = 0 \qquad \ldots(3)$$
$$B + 3D = 1 \qquad \ldots(4)$$
$$B + 3\frac{1}{2} = 1; \qquad C = 0$$
$$B + \frac{3}{2} = 1; \qquad A = 0$$
$$B = -\frac{1}{2}; \qquad D = +\frac{1}{2}$$

$$\therefore \qquad \int \frac{dt}{t^4 + 4t^2 + 3} = \int \left(\frac{-\frac{1}{2}}{(t^2+3)} + \frac{\frac{1}{2}}{t^2+2} \right) dt$$
$$= -\frac{1}{2\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} + \frac{1}{2} \tan^{-1} t + c$$

(Ans. $\mathbf{\frac{1}{2} \tan^{-1} t - \frac{1}{2\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} + c}$**)**

PROBLEM 6.42

Evaluate $\displaystyle\int \frac{8dx}{x^4 + 2x^3}$

Solution: $\displaystyle\int \frac{8dx}{x^4 + 2x^3} = 8\int \frac{dx}{x^3(x+2)}$ use integration by parts

$$\frac{1}{x^3(x+2)} = \frac{Ax^2 + Bx + C}{x^3} + \frac{D}{x+2}$$
$$\Rightarrow \qquad \frac{1}{x^3(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+2}$$
$$\Rightarrow \qquad \frac{1}{x^3(x+2)} = \frac{(Ax^2 + Bx + C)(x+2) + Dx^3}{x^3(x+2)}$$
$$1 = Ax^3 + 2Ax^2 + Bx^2 + 2Bx + Cx + 2C + Dx^3$$
$$1 = 2C \qquad \ldots(1)$$

$$\Rightarrow \qquad C = \frac{1}{2}$$

$$0 = A + D \qquad \ldots(2)$$

$$0 = 2A + B \qquad \ldots(3)$$

$$0 = 2B + C \qquad \ldots(4)$$

$$\Rightarrow \qquad 0 = 2B + \frac{1}{2}$$

$$2B = -\frac{1}{2} \qquad \therefore \quad B = -\frac{1}{4}$$

$$0 = 2A + B$$

$$0 = 2A - \frac{1}{4}$$

$$\therefore \qquad 2A = \frac{1}{4} \qquad \therefore \quad A = \frac{1}{8}$$

$$0 = A + D$$

$$0 = \frac{1}{8} + D \qquad \therefore \quad D = -\frac{1}{8}$$

$$\therefore \qquad \int \frac{8dx}{x^4 + 2x^3} = 8\int \left[\frac{\frac{1}{8}}{x} - \frac{\frac{1}{4}}{x^2} + \frac{\frac{1}{2}}{x^3} - \frac{\frac{1}{8}}{x+2} \right] dx$$

$$= \ln x + \frac{2}{x} - \frac{2}{x^2} - \ln(x+2) + c = \ln\frac{x}{x+2} + \frac{2}{x} - \frac{2}{x^2} + c$$

PROBLEM 6.43

Evaluate $\displaystyle\int \frac{\cos x\,dx}{\sqrt{1+\cos x}}$

Solution: $\displaystyle\int \frac{\cos x\,dx}{\sqrt{1+\cos x}}$

$\cos 2x = \cos^2 x - \sin^2 x$
$\therefore \cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$
$\cos^2 x + \sin^2 x = 1$
$\sin^2 x = 1 - \cos^2 x$

$$\Rightarrow \int \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\sqrt{1 + \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}} dx$$

$$\int \frac{\cos^2\frac{x}{2} - (1-\cos^2\frac{x}{2})}{\sqrt{1+\cos^2\frac{x}{2} - 1 - \cos^2\frac{x}{2}}}dx \Rightarrow \frac{2\cos^2\frac{x}{2} - 1}{\sqrt{2\cos^2\frac{x}{2}}}dx$$

$$\Rightarrow \frac{1}{\sqrt{2}}\int \frac{2\cos^2\frac{x}{2} - 1}{\cos\frac{x}{2}}dx$$

$$\Rightarrow \frac{1}{\sqrt{2}}\int \frac{2\cos^2\frac{x}{2}}{\cos\frac{x}{2}} - \frac{1}{\cos\frac{x}{2}}dx$$

$$\Rightarrow \frac{1}{\sqrt{2}}\int \left(2\cos\frac{x}{2} - \sec\frac{x}{2}\right)dx$$

As,

$$\int \cos\frac{x}{2}dx = \sin\frac{x}{2} + c$$

Let

$$u = \frac{x}{2} \qquad du = \frac{1}{2}$$

$$\Rightarrow \quad \frac{2}{\sqrt{2}}\int \left[2 \times \cos\frac{x}{2} \cdot \frac{1}{2}dx - 2\int \sec\frac{x}{2} \cdot \frac{1}{2}dx\right] = \frac{1}{\sqrt{2}}\left(4\sin\frac{x}{2} - 2\ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right| + c\right)$$

(Ans. $\sqrt{2}\left(2\sin\frac{x}{2} - \ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right|\right) + c$**)**

PROBLEM 6.44

Evaluate $\int \frac{x\,dx}{x + \sqrt{x} + 1}$

Solution: $\int \frac{x\,dx}{x + \sqrt{x} + 1}$

Let

$$x = y^2 \quad \Rightarrow \quad dx = 2y\,dy$$

$$\int \frac{x\,dx}{x+\sqrt{x}+1} = 2\int \frac{y^3\,dy}{y^2+y+1} = 2\int y - 1 + \frac{1}{y^2+y+1}\,dy$$

$$= 2\left(\frac{y^2}{2} - y\right) + 2\int \frac{dy}{y^2+y+1}$$

$$u^2 = \left(y+\frac{1}{2}\right)^2$$

$$u = y + \frac{1}{2}$$

$$a^2 = \frac{1}{4}$$

$$a = \frac{\sqrt{3}}{2}$$

$$y^2 = x$$

We write

$$y^2 + y + 1 \Rightarrow \left(y + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\therefore \qquad \int \frac{x\,dx}{x+\sqrt{x}+1} = 2\left(\frac{y^2}{2} - y + \frac{2}{\sqrt{3}}\tan^{-1}\frac{2y+1}{\sqrt{3}}\right) + c$$

$$= 2\frac{y^2}{2} - 2y + \frac{4}{\sqrt{3}}\tan^{-1}\frac{2y+1}{\sqrt{3}} + c$$

$$= 2x - 2\sqrt{x} + \frac{4}{\sqrt{3}}\tan^{-1}\frac{2\sqrt{x}+1}{\sqrt{3}} + c$$

(Ans. $\boldsymbol{x - 2\sqrt{x} + \frac{4}{\sqrt{3}}\tan^{-1}\frac{2\sqrt{x}+1}{\sqrt{3}} + c}$**)**

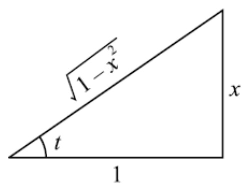
PROBLEM 6.45

Evaluate $\displaystyle\int \frac{dt}{\sec^2 t + \tan^2 t}$

Solution: $\displaystyle\int \frac{dt}{\sec^2 t + \tan^2 t}$

Let

$$x = \tan t \qquad dx = \sec^2 dt \qquad dt = \frac{dx}{1+x^2}$$



$$\int \frac{dt}{\sec^2 t + \tan^2 t} = \int \frac{\frac{dx}{1+x^2}}{1+x^2+x^2} = \int \frac{\frac{dx}{1+x^2}}{1+2x^2}$$

$$\int \frac{dx}{1+x^2} \cdot \frac{1}{1+2x^2} \Rightarrow \int \frac{dx}{(1+x^2)(1+2x^2)}$$

Integration by partial fraction

$$\frac{1}{(1+x^2)(1+2x^2)} = \frac{Ax+B}{1+x^2} + \frac{cx+D}{1+x^2}$$

$$\frac{1}{(1+x^2)(1+2x^2)} = \frac{(Ax+B)(1+2x^2)+(cx+D)(1+x^2)}{(1+x^2)(1+2x^2)}$$

$$1 = Ax + 2Ax^3 + B + 2Bx^2 + cx + cx^3 + D + Dx^2$$

$$1 = B + D \qquad \ldots(1)$$

$$0 = A + c \qquad \ldots(2)$$

$$0 = 2B + D \qquad \ldots(3)$$

$$0 = 2A + c \qquad \ldots(4)$$

Substitute Equation (1) into Equation (3):

$$1 = 1 = B + D$$

$$\therefore \quad B = 1 - D$$

$$0 = 2B + D$$

$$0 = 2(1-D) + D$$

$$0 = 2 - 2D + D$$

$$0 = 2 - D$$

$$\therefore \quad D = 2$$

$$\therefore \quad B = -1$$

$$A = -c$$

$$2(A) + c = 0$$

$$-2c + c = 0$$

$$\therefore \quad A = 0, \quad B = -1, \quad c = 0$$

$$-c = 0$$

$$D = 2$$

$$\therefore \quad c = 0$$

$$\therefore \quad A = 0$$

$$\int \frac{dx}{(1+x^2)(1+2x^2)} = \int \left[\frac{-1}{1+x^2} + \frac{2}{1+2x^2} \right] dx$$

$$[\text{As} \quad u^2 = 2x^2$$

$$u = \sqrt{2}x$$

$$du = \sqrt{2}$$

$$\int \frac{2}{1+2x^2}dx = \frac{2}{\sqrt{2}}\int \frac{\sqrt{2}}{1+2x^2}dx$$
$$= \frac{2}{\sqrt{2}}\tan^{-1}\sqrt{2}x + c = \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}}\tan^{-1}\sqrt{2}x$$
$$= \sqrt{2}\tan^{-1}\sqrt{2}x]$$
$$= -\tan^{-1}x + \sqrt{2}\tan^{-1}\sqrt{2x} + c$$
$$= -t + \sqrt{2}\tan^{-1}(\sqrt{2}\tan t) + c$$

(Ans. $\sqrt{2}\tan^{-1}(\sqrt{2}\tan t) - t + c$**)**

PROBLEM 6.46

Evaluate $\int \frac{dx}{1+\cos^2 x}$

Solution: $\int \frac{dx}{1+\cos^2 x}$

As $\cos^2 x = \frac{1+\cos 2x}{2}$

$$\int \frac{dx}{1+\cos^2 x}dx = \int \frac{1}{1+\cos^2 x+\cos^2 x-\cos^2 x}dx$$
$$= \int \frac{1}{2\cos^2 x+1-\cos^2 x}dx = \int \frac{1}{2\cos^2 x+\sin^2 x}dx$$
$$= \int \frac{1}{2\cos^2 x+\sin^2 x}\cdot \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}}dx$$
$$= \int \frac{1}{2\frac{\cos^2 x}{\cos^2 x}+\frac{\sin^2 x}{\sin^2 x}}\cdot \frac{1}{\cos^2 x}dx$$
$$= \int \frac{1}{2+\tan^2 x}\cdot \frac{1}{\cos^2 x}dx$$

Let $\quad u = \tan x$

$$du = \sec^2 x\,dx$$

$$= \int \frac{1}{2+u^2}\cdot du$$

$$= \int \frac{1}{(\sqrt{2})^2 + u^2} du$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + c$$

(Ans. $\mathbf{\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + c}$**)**

PROBLEM 6.47

Evaluate $\int \ln(\sqrt{x} + \sqrt{1+x})dx$

Solution: $\int \ln(\sqrt{x} + \sqrt{1+x})dx$

Use integration by partial fraction.

Let $\quad u = \ln(\sqrt{x} + \sqrt{1+x}) = \ln(1 + 2\sqrt{x})$

$$du = \frac{1}{(1+2\sqrt{x})} \cdot \frac{dx}{\sqrt{x}} = \frac{dx}{\sqrt{x} + 2x}$$

$$\int u\,dv = uv - \int v\,dv$$

$$\int \ln(\sqrt{x} + \sqrt{1+x})dx = x\ln(\sqrt{x} + \sqrt{1+x}) - \frac{1}{2}\int \frac{x\,dx}{2\sqrt{x^2+x}}$$

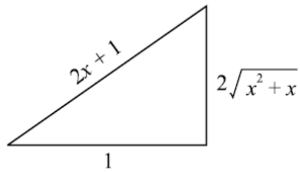
To find

$$\frac{1}{2}\int \frac{x\,dx}{2\sqrt{x^2+x}} = \frac{1}{2}\int \frac{x\,dx}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}} = \int \frac{x\,dx}{\sqrt{(2x+1)^2 - 1}},$$

let $2x + 1 = \sec\theta \Rightarrow 2\,dx = \sec\theta \cdot \tan\theta \cdot d\theta$

$$\frac{1}{2}\int \frac{x\,dx}{2\sqrt{x^2+x}} = \int \frac{\frac{\sec\theta-1}{2}\cdot\frac{\sec\theta\cdot\tan\theta\cdot d\theta}{2}}{\sqrt{\sec^2\theta-1}}$$

$$= \frac{1}{4}\int(\sec^2\theta - \sec\theta)d\theta = \frac{1}{4}\tan\theta - \ln|\sec\theta\cdot\tan\theta|$$



$$= \frac{1}{4}\left(2\sqrt{x^2+x} - \ln|2x+1+2\sqrt{x^2+x}|\right)$$

$$\int \ln(\sqrt{x}+\sqrt{1+x})dx = x\ln\left(\sqrt{x}+\sqrt{1+x}\right) - \frac{1}{2}\sqrt{x^2+x} + \frac{1}{4}\ln|2x+1+2\sqrt{x^2+x}|\Big) + c$$

(Ans. $\boldsymbol{x\ln(\sqrt{x}+\sqrt{1+x} - \frac{\sqrt{x^2+x}}{2} + \frac{1}{4}\ln\left|2x+1+2\sqrt{x^2+x}\right| + c}$**)**

PROBLEM 6.48

Evaluate $\int x\ln(x^3+x)dx$

Solution: $\int x\ln(x^3+x)dx$

Use integration by partial fraction

Let $\quad u = \ln(x^3+x) \Rightarrow du = \frac{3x^2+1}{x^3+x}dx$

$$dv = xdx \Rightarrow v = \frac{x^2}{2}$$

$$\therefore \quad \int x\ln(x^3+x)dx = \frac{x^2}{2}\ln(x^3+x) - \frac{1}{2}\int\frac{3x^3+x}{x^2+1}dx$$

$$= \frac{x^2}{2}\ln(x^3+x) - \frac{1}{2}\int 3x - \frac{2x}{x^2+1}dx$$

$$= \frac{x^2}{2}\ln(x^3 + x) - \frac{3}{4}x^2 + \frac{1}{2}\ln(x^2 + 1) + c$$

$$\textbf{(Ans. } \frac{x^2}{2}\ln(x^3 + x) - \frac{3}{4}x^2 + \frac{1}{2}\ln(x^2 + 1) + c\textbf{)}$$

PROBLEM 6.49

Evaluate $\int \frac{\cos x dx}{\sqrt{4-\cos^2 x}}$

Solution:

$$\int \frac{\cos x\, dx}{\sqrt{4-\cos^2 x}} = \int \frac{\cos x\, dx}{\sqrt{4-(1-\sin^2 x)}}$$

$$= \int \frac{\cos x\, dx}{\sqrt{4-1+\sin^2 x}} = \int \frac{\cos x\, dx}{\sqrt{3+\sin^2 x}}$$

Let $\sin x = \sqrt{3}\tan\theta$

$$\cos x\, dx = \sqrt{3}\sec^2\theta\, d\theta$$

$$\int \frac{\sqrt{3}\sec^2\theta\, d\theta}{\sqrt{3+3\tan^2\theta}} = \int \frac{\sqrt{3}\sec^2\theta\, d\theta}{\sqrt{3}\left(\sqrt{1+\tan^2\theta}\right)} = \frac{\sqrt{3}}{\sqrt{3}}\int \frac{\sec^2\theta}{(\sqrt{\sec^2\theta})}d\theta$$

$$\int \frac{\sec^2\theta d\theta}{\sec\theta} = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + c$$

$$= \ln\left|\frac{\sqrt{3+3\sin^2\theta}}{\sqrt{3}} + \frac{\sin x}{\sqrt{3}}\right| + c$$

$$\textbf{(Ans. } \ln\left|\frac{\sqrt{3+3\sin^2\theta}}{\sqrt{3}} + \frac{\sin x}{\sqrt{3}}\right| + c\textbf{)}$$

PROBLEM 6.50

Evaluate $\displaystyle\int \frac{\sec^2 x\,dx}{\sqrt{4-\sec^2 x}}$

Solution:

$$\int \frac{\sec^2 x\,dx}{\sqrt{4-\sec^2 x}} = \int \frac{dx}{\cos^2 x\sqrt{4-\sec^2 x}} = \int \frac{dx}{\cos x \cdot \cos x\sqrt{4-\sec^2 x}}$$

$$= \int \frac{dx}{\cos x \cdot \sqrt{\cos^2 x(4-\sec^2 x)}} = \int \frac{dx}{\cos x\sqrt{4\cos^2 x-1}}$$

$$= \int \frac{dx}{\sqrt{\cos^2 x} \cdot \sqrt{4\cos^2 x-1}}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \int \frac{dx}{\sqrt{\frac{1+\cos 2x}{2}} \cdot \sqrt{4\left(\frac{1+\cos 2x}{x}\right)-1}}$$

$$= \int \frac{dx}{\sqrt{\frac{1}{2}}\sqrt{1+\cos 2x} \cdot \sqrt{\frac{4+4\cos 2x-1}{2}}}$$

$$= \int \frac{dx}{\sqrt{\frac{1}{2}}\sqrt{1+\cos 2x} \cdot \sqrt{1+\cos 2x}}$$

$$= \sqrt{2}\int \frac{dx}{\sqrt{1+\cos 2x} \cdot \sqrt{1+2\cos 2x}}$$

where $\quad d\theta = \frac{2dz}{1+z^2}$

$$\cos\theta = \frac{1-z^2}{1+z^2}$$

$$= \frac{\sqrt{2}}{2} \int \frac{d\theta}{\sqrt{1+\cos\theta} \cdot \sqrt{1+2\cos\theta}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{\frac{2dz}{1+z^2}}{\sqrt{\left(1+\frac{1-z^2}{1+z^2}\right)} \cdot \sqrt{\left(1+2\frac{1-z^2}{1+z}\right)}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{\frac{dz}{1+z^2}}{\sqrt{\left(\frac{(1+z^2)+(1-z^2)}{1+z^2}\right) \cdot \left(\frac{(1+z^2)+2(1-z^2)}{1+z^2}\right)}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{\frac{2dz}{1+z^2}}{\sqrt{\frac{1+z^2+1-z^2}{1+z^2} \cdot \frac{1+z^2+2-z^2}{1+z^2}}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{\frac{dz}{1+z^2}}{\sqrt{\frac{2}{1+z^2} \cdot \frac{3-z^2}{1+z^2}}} = \frac{1}{\sqrt{2}} \int \frac{\frac{dz}{1+z^2}}{\sqrt{\frac{2(3-z^2)}{(1+z^2)^2}}}$$

$$= \frac{2}{\sqrt{2}} \int \frac{\frac{dz}{1+z^2}}{\sqrt{2}\,\frac{\sqrt{3-z^2}}{1+z^2}} = \int \frac{dz}{\sqrt{3-z^2}}$$

$$= \sin^{-1}\frac{z}{\sqrt{3}} + c = \sin^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\theta}{2}\right) + c$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{3}}\tan x\right) + c$$

(Ans. $\mathbf{\sin^{-1}\left(\frac{1}{\sqrt{3}}\tan x\right) + c}$**)**

PROBLEM 6.51

Evaluate $\displaystyle\int \frac{dt}{t-\sqrt{1-t^2}}$

Solution: $\displaystyle\int \frac{dt}{t-\sqrt{1-t^2}}$

Let $$t=\sin\theta \quad \Rightarrow \quad dt=\cos\theta\, d\theta$$

where $$\cos\theta=\frac{1-z^2}{1+z^2}$$

$$d\theta=\frac{2dz}{1+z^2}$$

$$\sin\theta=\frac{2z}{1+z^2}$$

$$\int\frac{\cos\theta\, d\theta}{\sin\theta-\cos\theta}=\int\frac{\frac{1-z^2}{1+z^2}\cdot\frac{2dz}{1+z^2}}{\frac{2z}{1+z^2}-\frac{1-z^2}{1+z^2}}$$

$$=\int\frac{\frac{2(1-z^2)}{(1+z^2)^2}}{\frac{2z-1+z^2}{(1+z^2)}}=\int\frac{2(1-z^2)}{(1+z^2)(z^2+2z-1)}$$

$$=2\int\frac{(1-z^2)}{(1+z^2)(z^2+2z-1)}$$

Use integration by partial fraction.

$$\frac{(1-z^2)}{(1+z^2)(z^2+2z-1)}=\frac{Az+B}{z^2+1}+\frac{Cz+D}{z^2+2z-1}$$

$$\frac{(1-z^2)}{(1+z^2)(z^2+2z-1)}=\frac{(Az+B)(z^2+2z-1)+(cz+D)(z^2+1)}{(1+z^2)(z^2+2z-1)}$$

$$1-z^2=Az^3+2Az^2-Az+Bz^2+2Bz-B+Cz^3+Cz+Dz^2+D$$

$$0 = A + C \quad \ldots(1)$$

$$-1 = 2A + B + D \quad \ldots(2)$$

$$1 = -B + D \quad \ldots(3)$$

$$0 = 2B - A + C \quad \ldots(4)$$

$$\Rightarrow \quad A = -\frac{1}{2}, \quad B = -\frac{1}{2}$$

$$\Rightarrow \quad C = \frac{1}{2}, \quad D = \frac{1}{2}$$

$$= 2\int \frac{1 - z^2 dz}{(1+z^2)(z^2+2z-1)} = \int \frac{-z-1}{z^2+1} + \frac{z+1}{z^2+2z-1} dz$$

$$= \int \frac{-z}{z^2+1} - \frac{1}{z^2+1} + \frac{z+1}{z^2+2z-1} dz$$

$$= \frac{1}{2}\ln\left|z^2+1\right| - \tan^{-1} z + \frac{1}{2}\ln\left|z^2+2z-1\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{z^2+2z-1}{z^2+1}\right| - \tan^{-1} z + c$$

$$d\theta = \frac{2dz}{1+z^2}$$

$$\theta = 2\tan z$$

$$\tan^{-1} z = \frac{\theta}{2}$$

$$z = \tan\frac{\theta}{2} = \frac{1}{2}\ln\left|\frac{\tan^2\frac{\theta}{2} + 2\tan\frac{\theta}{2} - 1}{\tan^2\frac{\theta}{2} + 1}\right| - \frac{\theta}{2} + c$$

$$= \frac{1}{2}\ln\left|\frac{\dfrac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} + 2\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} - 1}{\dfrac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} + 1}\right| - \frac{\theta}{2} + c$$

$$= \frac{1}{2} \ln \left| \frac{\dfrac{\sin^2 \dfrac{\theta}{2} + 2\sin \dfrac{\theta}{2} - \cos^2 \dfrac{\theta}{2}}{\cos^2 \dfrac{\theta}{2}}}{\dfrac{\sin^2 \dfrac{\theta}{2} + \cos^2 \dfrac{\theta}{2}}{\cos^2 \dfrac{\theta}{2}}} \right| - \frac{\theta}{2} + c$$

$$= \frac{1}{2} \ln \left| \frac{\sin^2 \dfrac{\theta}{2} + \sin \dfrac{\theta}{2} - \cos^2 \dfrac{\theta}{2}}{\cos^2 \dfrac{\theta}{2} + \cos^2 \dfrac{\theta}{2}} \right| - \frac{\theta}{2} + c$$

$$\sin^2 \theta = \frac{1 - 2\cos\theta}{2} \qquad \sin^2 \theta = \frac{1 - \cos\theta}{2}$$

$$= \frac{1}{2} \ln \left| \frac{\dfrac{1 - \cos\theta}{2} + \sin\theta - \dfrac{1 + \cos\theta}{2}}{1} \right| - \frac{\theta}{2} + c$$

$\because$ $$2\sin\theta\cos\theta = \sin 2\theta$$

$$2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta$$

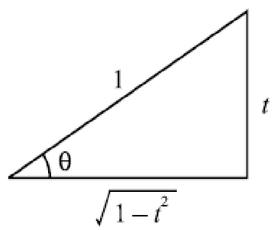
$$\cos^2 \frac{\theta}{2} = \frac{1 + \cos\theta}{2}$$

$$= \frac{1}{2} \ln \left| \frac{1 - \cos\theta + 2\sin\theta - 1 - \cos\theta}{2} \right| - \frac{\theta}{2} + c$$

$$= \frac{1}{2} \ln \left| \frac{-2\cos\theta + 2\sin\theta}{2} \right| - \frac{\theta}{2} + c$$

$$= \frac{1}{2} \ln \left| \frac{2(\sin\theta - \cos\theta)}{2} \right| - \frac{\theta}{2} + c$$

$$= \frac{1}{2} \ln |\sin\theta - \cos\theta| - \frac{\theta}{2} + c$$



$$= \frac{1}{2} \ln \left| t - \sqrt{1 - t^2} \right| - \frac{1}{2} \sin^{-1} t + c$$

As, $\sin\theta = t$

$$\cos\theta = \sqrt{1 - t^2}$$

(Ans. $\mathbf{\frac{1}{2} \ln(t - \sqrt{1 - t^2}) - \frac{1}{2} \sin^{-1} t + c}$)

PROBLEM 6.52

Evaluate $\int e^{-x} \cdot \tan^{-1} e^x dx$

Solution: $\int e^{-x} \cdot \tan^{-1} e^x dx$

Integration by part

Let

$$u = \tan^{-1} e^x \quad \Rightarrow \quad du = \frac{e^x}{1+e^{2x}} dx$$

$$dv = e^{-x} dx \quad \Rightarrow \quad v = -e^{-x}$$

$$\int e^{-x} \tan^{-1} e^x dx = -e^{-x} \tan^{-1} e^x + \int \frac{1}{1+e^{2x}} dx$$

Let $\qquad e^x = \tan\theta \quad \Rightarrow \quad e^x\, dx = \sec^2\theta\, d\theta$

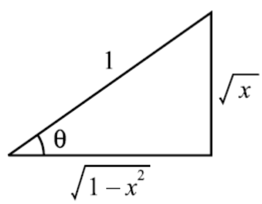
$$dx = \frac{\sec^2\theta\, d\theta}{\tan\theta}$$

To find $\int \frac{1}{1+e^{2x}} dx$

$$= \int \frac{\frac{\sec^2\theta\, d\theta}{\tan\theta}}{1+\tan^2\theta} = \int \frac{\frac{\frac{\sin^2\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}}{1+\tan^2\theta} d\theta = \int \frac{\tan\theta}{1+\tan^2\theta} d\theta = \int \frac{\frac{\sin\theta}{\cos\theta}}{\sec^2\theta} d\theta$$

$$= \int \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}} d\theta = \int \frac{\cos}{\sin\theta} d\theta = \int \cot\theta \cdot d\theta = \ln\ \sin\theta + c$$

$$\sin\theta = \frac{e^x}{\sqrt{1+e^{2x}}}$$



$\therefore \quad \ln|\sin\theta| + c$

$$\Rightarrow \ln\left|\frac{e^x}{\sqrt{1+e^{2x}}}\right| \Rightarrow \ln e^x - \ln\sqrt{1+e^{2x}}$$

$$x-\ln(1+e^{2x})^{\frac{1}{2}}$$

$$x-\frac{1}{2}\ln|1+e^{2x}|$$

$$\therefore \quad \int e^{-x}\tan^{-1}e^{x}dx = -e^{-x}\tan^{-1}e^{x}+x-\frac{1}{2}\ln|1+e^{2x}|+c$$

(Ans. $-e^{-x}\cdot\tan^{-1}e^{x}+x-\frac{1}{2}\ln(1+e^{2x})+c$**)**

PROBLEM 6.53

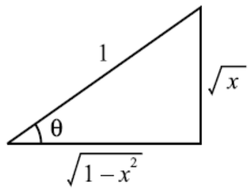
Evaluate $\int \sin^{-1}\sqrt{x}\,dx$

Solution: $\int \sin^{-1}\sqrt{x}\,dx$

Integration by parts

Let $\quad u=\sin^{-1}\sqrt{x} \qquad du=\dfrac{dx}{2\sqrt{x}\sqrt{1-x}}$

$$dv=dx \quad\Rightarrow\quad v=x$$



$$\therefore \quad \int \sin^{-1}\sqrt{x}\,dx = x\sin^{-1}\sqrt{x}-\frac{1}{2}\int\frac{\sqrt{x}}{\sqrt{1-x}}dx$$

To find $\dfrac{1}{2}\displaystyle\int\frac{\sqrt{x}}{\sqrt{1-x}}dx$

let $\quad x=\sin^2\theta$

$$dx=2\sin\theta\cos\theta\,d\theta$$

$$\frac{1}{2}\int\frac{\sqrt{x}}{\sqrt{1-x}}dx=\frac{1}{2}\int\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\cdot 2\sin\theta\cos\theta\,d\theta$$

$$=\int\sin^2\theta\,d\theta=\int\frac{1-\cos 2\theta}{2}d\theta=\frac{1}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right)+c$$

$$x=\sin^2\theta$$

$$\sqrt{x}=\sin\theta$$

$$\therefore \quad \theta=\sin^{-1}\sqrt{x}$$

$$\Rightarrow \frac{1}{2}\left(\sin^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x}\right)$$

$$\therefore \qquad \int \sin^{-1}\sqrt{x}\,dx = x\sin^{-1}\sqrt{x} - \frac{1}{2}\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x}\sqrt{1-x} + c$$

(Ans. $\boldsymbol{x\sin^{-1}\sqrt{x} - \frac{1}{2}\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x-x^2} + c}$**)**

PROBLEM 6.54

Evaluate $\boldsymbol{\int \frac{\cos 2x - 1}{\cos 2x + 1}dx}$

Solution: $\int \frac{\cos 2x - 1}{\cos 2x + 1}dx$

$$-\int \frac{\frac{1-\cos 2x}{2}}{\frac{1+\cos 2x}{2}}dx = -\int \frac{\sin^2 x}{\cos^2 x}dx$$

$$= -\int \tan^2 dx = -\int (\sec^2 - 1)dx$$

$$= -(\tan x - x) + c$$

$$= x - \tan x + c$$

(Ans. $\boldsymbol{x - \tan x + c}$**)**

CHAPTER 7

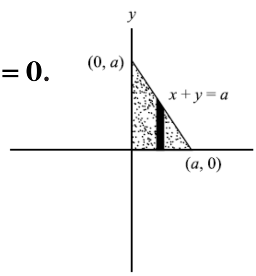
APPLICATION OF INTEGRALS

PROBLEMS

PROBLEM 7.1

Find the area of the region bounded by the given curves and lines for the following problems:

1. **The coordinate axes and the line $x+y=a$**
2. **The x-axis and the curve $y=e^x$ and the lines $x=0, x=1$**
3. **The curve $y^2+x=0$ and the line $y=x+2$**
4. **The curves $x=y^2$ and $x=2y-y^2$**
5. **The parabola $x=y-y^2$ and the line $x+y=0$.**



Solution:

1. The coordinate axes and the line $x+y=a$

$$\left.\begin{matrix} x+y=a \\ x=0 \\ (0,a) \end{matrix}\right\} \quad \left.\begin{matrix} y=0 \\ x+y=a \\ \therefore x=a \\ (a,0) \end{matrix}\right\}$$

$$x+y=a$$

$(0, a), (a, 0)$

$$\int_{x_1=0}^{x_2=a} dx$$

$$x_1 = 0 \quad \Delta x = x_2 - x_1$$
$$x_2 = a$$

$$\left.\begin{matrix} y_1 = 0 \\ y_2 = a \end{matrix}\right\} \Rightarrow$$

$$x + y = a$$

$\therefore$ $$y = a - x$$

$$\int_0^{a-x} dy \quad \therefore$$

$$\int_0^a \int_0^{a-x} dy\, dx$$

$$A = \int_0^a \int_0^{a-x} dy\, dx = \int_0^a (4)_0^{a-x}\, dx = \int_0^a (a-x)\, dx$$

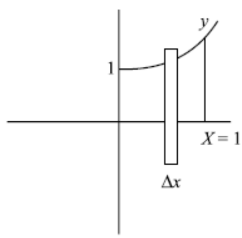
$\Rightarrow$ $$= \left(ax - \frac{x^2}{2}\right)_0^a$$

$$= \left(a^2 - \frac{a^2}{2}\right) - 0 = \frac{a^2}{2}$$

2. The x-axis and the curve $y = e^x$ and the lines $x = 0, x = 1$

$$x = 1, x = 0$$
$$y = e^x$$

x	$y = e^x$
0	1
1	2.78
1.5	4.48



$$A = \int_0^1\int_0^{e^x} dy\,dx \int_0^1 y\Big]_0^{e^x} dx = \int_0^1 (e^x - 0)dx = \int_0^1 e^x\,dx$$

$$= e^x\Big]_0^1 = e^1 - e^0 = e^1 - 1 = e - 1.$$

3. The curve $y^2 + x = 0$ and the line $y = x + 2$

$$y = x + 2 \text{ at } x = -1$$

$$y = -1 + 2 \quad \Rightarrow \quad y = 1$$

$$y = x + 2 \quad \Rightarrow \quad \text{at} \quad x = -4$$

$$y = -4 + 2 \quad \Rightarrow \quad y = -2$$

$\therefore$ $(-1, 1), (-4, -2)$

4. The curves $x = y^2$ and $x = 2y - y^2$

$$x = y^2 \qquad \ldots(1)$$

$$x = 2y - y^2 \qquad \ldots(2)$$

$$2y - y^2 = y^2$$

$$2y = 2y^2$$

$$y = y^2$$

$$y = 0, \quad y = 1$$

$$\left.\begin{array}{l} \therefore x = y^2 \\ \text{at } y = 0 \Rightarrow x = 0 \\ \text{at } y = 1 \Rightarrow x = 1 \end{array}\right\} \qquad \therefore (0, 0), (1, 1)$$

x	$x = y^2$
0	(0, 0)
0.5	$0.25 \Rightarrow (0.125, 0.5)$
0.7	$0.49 \Rightarrow (0.49, 0.7)$

$x = 2y - y^2$	y
0	0
1	1
0.75	0.5
0.91	0.75
0.96	1.2

$$A = \int_0^1 \int_{y^2}^{2y-y^2} dx\, dy$$

$$= \int_0^1 x \Big]_{y^2}^{2y-y^2} dy = \int_0^1 (2y - y^2 - y^2)\, dy = \int_0^1 2y - y^2\, dy$$

$$= \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3} - 0\right) = \frac{1}{3}$$

5. The parabola $x = y - y^2$ and the line $x + y = 0$

$$x = y - y^2 \qquad \ldots(1)$$

$$x + y = 0 \qquad \ldots(2)$$

$$x + y = 0$$

$$x = -y$$

$$-y = y - y^2$$

$$-2y = -y^2$$

$$2y = y^2$$

$$\left.\begin{matrix} y = 0 \\ y = 2 \end{matrix}\right\} \Rightarrow (1)$$

$$x = y - y^2 \text{ at } y = 0,\ x = 0 \text{ at } y = 2,\ x = -2$$

$$(0, 0)(-2, +2)$$

$$x + y = 0$$

$$x + y = 0 \quad \Rightarrow \quad y = -x$$

$$x = y - y^2$$

x	$y=-x$
–1	+1
–2	+2
–3	+3
0	0
1	–1
2	–2
3	–3

y	$x=y-y^2$
0	0
1	0
2	–2
0.5	+0.15

$$A=\int_0^2\int_{-y}^{2y-y^2}dx\,dy=\int_0^2 x\Big]_{-y}^{y-y^2}dy=\int_0^2(2y-y^2)dy$$

$$=y^2-\frac{y^3}{3}\Big]_0^2=4-\frac{8}{3}-0=\frac{4}{3}$$

(Ans. 1. $\frac{a^2}{2}$**; 2.** $e-1$**; 3.** $\frac{9}{2}$**; 4.** $\frac{1}{3}$**; 5.** $\frac{4}{3}$**)**

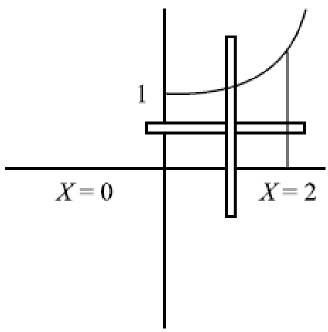
PROBLEM 7.2

Write an equivalent double integral with order of integration reversed for each integrals check your answer by evaluation both double integrals, and sketch the region.

1. $\int_0^2\int_1^{e^x}dy\,dx$

2. $\int_0^1\int_{\sqrt{y}}^1 dx\,dy$

3. $\int_0^{\sqrt{2}}\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\,dx\,dy$



Solution:

1.

$$\int_0^2\int_1^{e^x} dy\,dx = \int_0^2 y\Big]_1^{e^x} dx$$

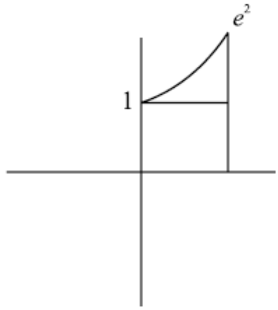
$$= \int_0^2 (e^x - 1)\,dx$$

$$= \left(e^x - x\right)_0^2 \Rightarrow (e^2 - 2) = (e^0 - 0)$$

$$= e^2 - 2 - e^0 - 0 = e^2 - 2 - 1 = e^2 - 3$$

$$y = e^x$$

x	$y = e^x$
0	1
1	27.8
1.5	4.48



$$x = 2, \quad x = 2, \quad e^2 = 7.389$$

$$y = e^x \Rightarrow \quad x = \ln y$$

$$y = e^0, \quad y = e^2$$

$$\int_1^{e^2}\int_{\ln y}^{2} dx\,dy = \int_1^{e^2} (2 - \ln y)\,dy = \int_1^{e^2} 2\,dy - \int_1^{e^2} \ln y\,dy$$

To find $\int_1^{e^2} \ln y\,dy$ use integration by part

$$\left.\begin{aligned} &\text{Let } u = \ln y \Rightarrow du = \frac{1}{y}dy \\ &dv = dy \Rightarrow v = y \end{aligned}\right\}$$

$\therefore$
$$\int \ln y\,dy = y\ln y - \int_{e^2} dy = y\ln y - y$$

$\therefore$
$$\int_1^{e^2} 2\,dy = y\ln y - \int_1^{e^2} \ln y\,dy$$

$$\Rightarrow \qquad [2y - y\ln y + y]_1^{e^2} = (2e^2 - e^2\ln e^2 + e^2) - (2 - ln1 + 1)$$

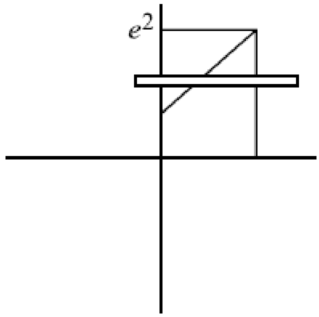
$$= (3e^2 - e^2\ln e^2) - (3 - \ln 1)$$

$$= 3e^2 - e^2\ln e^2 - 3$$

$$= 3e^2 - 2e^2 - 3$$

$$= e^2 - 3$$

y	$x = \ln y$
1	0
2	0.69
3	1.09
4	1.38
.	.
.	.
739	2



2. $$\int_0^1 \int_{\sqrt{y}}^1 dx\,dy$$

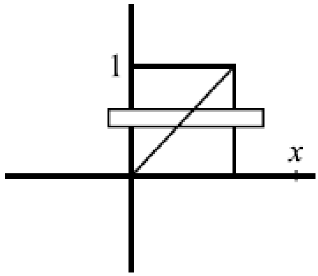
$$\int \sqrt{y}\,dy = \int (y)^{\frac{1}{2}}\,dy$$

$$\frac{(y)^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{y^{3/2}}{\frac{3}{2}} = \frac{\sqrt{y^3}}{\frac{3}{2}} = \frac{2}{3}\sqrt{y}$$

$$\int_0^1 (1 - \sqrt{y})\,dy$$

$$= {}_0^1\left| y - \frac{2}{3}\sqrt{y^3} \right|_0^1 = 1 - \frac{2}{3} - 0 = \frac{1}{3}$$

y	$x=\sqrt{y}$
0	0
0.5	0.707
0.75	0.866
1	1

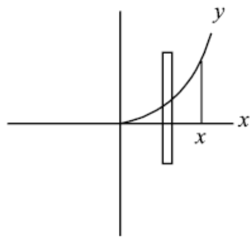


$$x=\sqrt{y} \Rightarrow y=x^2$$

$$y=0, \quad y=1$$
$$x=0, \quad x=1$$

$$\int_0^1 \int_0^{y=x^2} dy\,dx$$

x	$y=x^2$
0	0
1	1
2	4
3	9



$$\int_0^1 \int_0^{y=x^2} dy\,dx = \int_0^1 x^2\,dx$$

$$= \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}(1-0) = \frac{1}{3}$$

(Ans. $\displaystyle\int_0^1\int_0^{x^2} dy\,dx; \frac{1}{3}$**)**

3. $\displaystyle\int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\,dx\,dy$

$$\int_0^{\sqrt{2}} y\left[\sqrt{4-2y^2} - \left(-\sqrt{4-2y^2}\right)\right] dy$$

$$\int_0^{\sqrt{2}} y\left[(4-2y^2)^{\frac{1}{2}} + (4-2y^2)^{\frac{1}{2}}\right] dy$$

$$= \int_0^{\sqrt{2}} y\left(2(4-2y^2)^{\frac{1}{2}}\right)dy$$

$$= \int_0^{\sqrt{2}} 2y(4-2y^2)^{\frac{1}{2}}\,dy$$

$$= -\frac{1}{2}\int_0^{\sqrt{2}} -4y(4-2y^2)^{\frac{1}{2}}\,dy \Rightarrow -\frac{1}{2}\frac{(4-2y^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{\sqrt{2}}$$

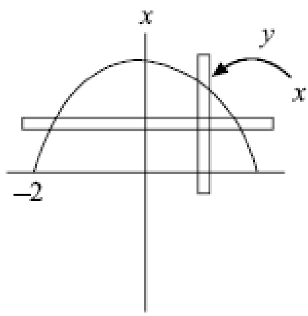
$$= -\frac{1}{3}(0-8) = \frac{8}{3}$$

$$\int_0^{\sqrt{2}}\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\,dx\,dy$$

$y = 0, \quad y = \sqrt{2}$

$x = \mp\sqrt{4-2y^2}$

$x = -\sqrt{4-2y^2}$	y
2	0
1.4	1
0	$\sqrt{2}$
–2	0
–1.4	1
0	$\sqrt{2}$



$$x = \mp\sqrt{4-2y^2} \Rightarrow y = \mp\frac{\sqrt{4-x^2}}{2}$$

$$y = \frac{\sqrt{4-x^2}}{2}$$

$$= \int_{-2}^{2} \int_{0}^{\frac{\sqrt{4-x^2}}{2}} y\,dy\,dx$$

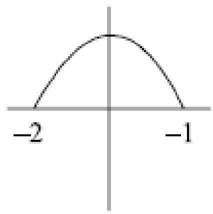
$$= \int_{-2}^{2} \frac{y^2}{0}\Bigg|_{0}^{\frac{\sqrt{4-x^2}}{2}} dx$$

$$= \frac{1}{2}\int_{-2}^{2} y\left(\frac{4-x^2}{2} - 0\right)dx$$

$$= \frac{1}{4}\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{1}{4}\left[8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right)\right] = \frac{8}{3}$$

x	$y = \frac{\sqrt{4-x^2}}{2}$
0	$\sqrt{2}$
1	1.22
2	0
–2	0
–1	1.22



(Ans. $\int_{-2}^{2} \int_{0}^{\frac{\sqrt{4-x^2}}{2}} y\,dy\,dx; \frac{8}{3}$**)**

PROBLEM 7.3

Find the volume of the tetrahedron bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the coordinate planes.

Solution:

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

$$V=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}\int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\,dy\,dx$$

$$=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} z\Big|_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dy\,dx$$

$$=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx=c\int_0^a\left[\left(1-\frac{x}{a}\right)y-\frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$=c\int_0^a\left[\left(1-\frac{x}{a}\right)b\left(1-\frac{x}{a}\right)-\frac{b^2\left(1-\frac{x}{a}\right)^2}{2b}\right]dx=\frac{bc}{2}\int_0^a\left(1-\frac{x}{a}\right)^2 dx$$

$$=\frac{-abc}{2}\frac{\left(1-\frac{x}{a}\right)^3}{3}\Bigg|_0^a=\frac{-abc}{2}(0-1)=\frac{1}{6}|abc|$$

PROBLEM 7.4

Find the volume bounded by the plane $z=0$ laterally by the elliptic cylinder $x^2+4y^2=4$ and above by the plane $z=x+2$

Solution:

$$x^2+4y^2=4$$

$$z=x+2 \Rightarrow= z=0$$

$$x^2+4y^2=4,\ x^2=4 \Rightarrow x=\pm x=\pm 2$$

$$x^2+4y^2=4$$

$$4y^2=4-x^2$$

$$y^2 = \frac{4-x^2}{4}$$

$$y = \pm\sqrt{\frac{4-x^2}{4}} = \pm\frac{1}{2}\sqrt{4-x^2}$$

$$\therefore \qquad \int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} = \int_{z=2}^{z=x+2}$$

$$V = \int_{-2}^{2}\int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}}\int_{0}^{x+2} dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} (x+2)\,dy\,dx$$

$$= \int_{-2}^{2}(x+2) \quad y\,\Big|_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} \; dx = \int_{-2}^{2}(x+2)\left[\sqrt{\frac{4-x^2}{4}} + \sqrt{\frac{4-x^2}{4}}\right]dx$$

$$= \int_{-2}^{2}(x+2) \quad \sqrt{4-x^2}\; dx = \text{let}\, x = 2\sin\theta \Rightarrow dx = 2\cos\theta\, d\theta$$

$$\therefore \qquad \theta = \sin^{-1}\frac{x}{2} \xrightarrow{at\ x=2} \theta = \frac{\pi}{2} \xrightarrow{at\ x=2}$$

$$\xrightarrow{at\ x=2} \theta = -\frac{\pi}{2}$$

$$\therefore \qquad V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\sin\theta + 2)\ \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\, d\theta$$

$$= 8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\cos^2\theta + \cos^2\theta\, d\theta$$

$$= 8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\cos^2\theta + \frac{1+\cos\theta}{2}\, d\theta$$

$$= 8\left[\frac{-\cos^3\theta}{3} + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 8\left[-\frac{1}{3}(0-0) + \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + \frac{1}{4}(0-0)\right] = 4\pi$$

(Ans. 4π)

PROBLEM 7.5

Find the lengthed of the following curves:

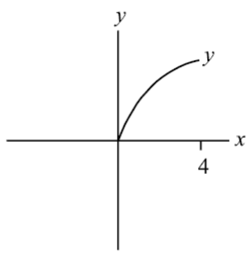
1. $y = x^{\frac{3}{2}}$ **from (0, 0) to (4, 8)**
2. $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$ **from $x = 1$ to $x = 3$**
3. $x = \dfrac{y^4}{4} + \dfrac{1}{8y^2}$ **from $y = 1$ to $y = 2$**
4. $(y+1)^2 = 4x^3$ **from $x = 0$ to $x = 1$**

Solution:

1. $y = x^{3/2}$ from $(0, 0)$ to $(4, 8)$

$ds^2 = dx^2 + dy^2$ where ds : differential of arc length

$$y = x^{\frac{3}{2}} \Rightarrow \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$$



$\therefore$ $$dy = \frac{3}{2}x^{\frac{1}{2}}\,dx$$

$$dy^2 = \frac{9}{4}x\,dx^2$$

$\therefore$ $$ds^2 = dx^2 + dy^2$$

$$= dx^2 + \left(\frac{9}{4}x\,dx^2\right)$$

$$ds^2 = \left(1 + \frac{9}{4}x\,dx^2\right)$$

$$L = \therefore \quad ds = \sqrt{1 + \frac{9}{4}x}\,dx$$

∴ The position of the curve between the origin and point (4, 8)

∴ $$L = \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx$$

$$= \int_0^4 \left(1+\frac{9}{4}x\right)^{\frac{1}{2}} dx$$

$$= \int_0^4 \left(\frac{4+9x}{4}\right)^{\frac{1}{2}} dx$$

$$= \int_0^4 \left(\frac{4+9x}{\sqrt{4}}\right)^{\frac{1}{2}} dx$$

$$= \int_0^4 \left(\frac{4+9x}{2}\right)^{\frac{1}{2}} dx$$

$$= \frac{1}{2}\int_0^4 (4+9x)^{\frac{1}{2}} dx$$

$$= \frac{1}{2\times 9}\int_0^4 (4+9x)^{\frac{1}{2}} \cdot 9\,dx$$

$$= \frac{1}{18}\int_0^4 (4+9x)^{\frac{1}{2}} 9\,dx$$

$$= \frac{1}{18}\left[\frac{(4+9x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 = \frac{1}{27}\left[(4+9x)^{\frac{3}{2}}\right]_0^4$$

$$= \frac{1}{27}(4+9x)^{\frac{3}{2}} - \frac{3}{4^2} \Rightarrow \frac{1}{27}\left(\sqrt{40^3}-\sqrt{4^3}\right)$$

$$= \frac{1}{27}(80\sqrt{10}-8)$$

$$= \frac{8}{27}(10\sqrt{10}-1)$$

(Ans. $\frac{8}{27}(10\sqrt{10}-1)$)

2. $y=\frac{x^3}{3}+\frac{1}{4x}$ from $x=1$ to $x=3$.

$$\frac{dy}{dx}=\frac{1}{3}\cdot 3x^2-\frac{1}{4x^2}$$

$$\therefore \qquad dy=\left(x^2-\frac{1}{4x^2}\right)dx$$

$$ds^2=dx^2+dy^2$$

$$=dx^2+\left(x^2-\frac{1}{4x^2}\right)^2dx^2=\left[1+\left(x^2-\frac{1}{4x^2}\right)^2\right]dx$$

$$L=\int_1^3\sqrt{1+\left(x^2-\frac{1}{4x^2}\right)^2}\,dx$$

$$=\int_1^3\sqrt{1+\left(x^4-2x^2\frac{1}{4x^2}+\frac{1}{16x^4}\right)}dx$$

$$=\int_1^3\sqrt{\frac{16x^4+8x^4+1}{16x^4}}\,dx$$

$$\therefore \qquad L=\int_1^3\sqrt{\frac{16x^8+8x^4+1}{4x^4}}\,dx$$

$$L=\frac{1}{4}\int_1^3\sqrt{\frac{16x^8+8x^4+1}{x^2}}\,dx$$

$$=\frac{1}{4}\int_1^3\sqrt{\frac{(4x^4+1)^2}{x^2}}$$

$$=\frac{1}{4}\int_1^3\frac{(4x^4+1)^2}{x^2}$$

$$=\frac{1}{4}\int_1^3\left(4x^2+\frac{1}{x^2}\right)dx=\frac{1}{4}\quad\left[\frac{4}{3}x^3-\frac{1}{x}\right]_1^3$$

$$16x^8+8x^4+1$$

$$=(4x^4+1)^2$$

$$=\frac{1}{4}\left[36-\frac{1}{3}-\frac{4}{3}+1\right]=\frac{53}{6}$$

(Ans. $\frac{53}{6}$)

3. $x = \frac{y^4}{4} + \frac{1}{8y^2}$ from $y = 1$ to $y = 2$

$$\frac{dy}{dx} = \frac{4}{4} y^3 - \frac{1}{4y^3}$$

$$dx = \left(y^3 - \frac{1}{4y^3} \right)^2 dy^2 \Rightarrow ds^2 = dx^2 + dy^2$$

$$\therefore \quad L = \sqrt{1 + \left(y^6 - 2y^3 - \frac{1}{4y^3} + \frac{1}{16y^6} \right)}\, dy$$

$$= \int_1^2 \sqrt{1 + \left(y^6 - 2y^3 - \frac{1}{4y^3} + \frac{1}{16y^6} \right)}\, dy$$

$$= \int_1^2 \sqrt{y^6 + \frac{1}{2} + \frac{1}{16y^6}}\, dy$$

$$= \int_1^2 \sqrt{\frac{16y^{12} + 8y^6 + 1}{16y^6}}\, dy$$

$$= \int_1^2 \sqrt{\frac{16y^{12} + 8y^6 + 1}{16y^6}}\, dy$$

$$= \int_1^2 \sqrt{\frac{16y^{12} + 8y^6 + 1}{(y^3)^2}}\, dy$$

$$= \frac{1}{4} \int_1^2 \sqrt{\frac{16y^{12} + 8y^6 + 1}{(y^3)^2}}\, dy \Rightarrow \frac{1}{4} \int_1^2 \sqrt{\frac{(4y^6 + 1)^2}{(y^3)^2}}\, dy$$

$$= \frac{1}{4} \int_1^2 \frac{4y^6 + 1}{y^3}\, dy = \int_1^2 \left(4y^3 + \frac{1}{y^3} \right) dy$$

$$= \frac{1}{4} \left[y^4 - \frac{1}{2y^2} \right]_1^2$$

$$= \frac{1}{4} \left[16 - \frac{1}{18} - 1 + \frac{1}{2} \right] = \frac{123}{32}$$

(Ans. $\frac{123}{32}$)

4. $(y+1)^2 = 4x^3$ from $x=0$ to $x=1$.

$$y+1 = 2\sqrt{x^3}$$

$$y = \mp 2\sqrt{x^3} - 1 \Rightarrow y = 2x^{\frac{3}{2}} - 1$$

$$\frac{dy}{dx} = 2\times\frac{3}{2}x^{\frac{1}{2}} = 3\sqrt{x}$$

$$dy = 3\sqrt{x}\,dx$$

$$dy^2 = 9x\,dx^2$$

$$ds^2 = dx^2 + dy^2$$

$$= dx^2 + 9x\,dx^2$$

$$ds^2 = (1+9x)dx^2$$

$$ds = L$$

$$L = 2\int_0^1 \sqrt{1+9x}dx$$

$$= \frac{2}{9}\int_0^1 (1+9x)^{\frac{1}{2}}\cdot 9dx$$

$$= \frac{2}{9}\frac{(1+9x)^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= \frac{4}{27}\sqrt{(1+9x)^3}\Big|_0^1$$

$$= \frac{4}{27}[10\sqrt{10} - 1]$$

(Ans. $\mathbf{\frac{4}{27}(10\sqrt{10} - 1)}$**)**

PROBLEM 7.6

Find the distance travelled by the particle $P(x, y)$ between $t = 0$ and $t = 4$ if the position at time t is given by : $x = \frac{t^2}{2}; y = \frac{1}{3}(2t+1)^{3/2}$

Solution:

$$x = \frac{t^2}{2}, \quad y = \frac{1}{3} \quad (2t+1)^{\frac{3}{2}}$$

$$\frac{dx}{dt} = \frac{1}{2} \cdot 2t = t$$

$$y = \frac{1}{3}(2t+1)^{3/2} \Rightarrow \frac{dy}{dt} = \frac{1}{3} \cdot \frac{3}{2}(2t+1)^{\frac{1}{2}} \cdot 2$$

$$= (2t+1)^{\frac{1}{2}} = \sqrt{2t+1}$$

$$\therefore \qquad L = \int_0^4 \sqrt{dx^2 + dy^2}\, dy$$

$$dx = t\ dt \Rightarrow dx^2 = t^2 dt^2$$

$$dy = \sqrt{2t+1dt} \Rightarrow dy^2 = (2t+1)dt$$

$$\therefore \qquad L = \int_0^4 \sqrt{\left(t^2 + 2t + 1\right) dt^2} = \int_0^4 \sqrt{t^2 + 2t + 1}\, dt$$

$$= \int_0^4 \sqrt{(t+1)^2}\, dt = \int_0^4 (t+1)^2\, dt$$

$$= \frac{t^2}{2} + t\Big|_0^4 = 8 + 4 - 0 = 12$$

(Ans. 12)

PROBLEM 7.7

The position of a particle $P(x, y)$ at time t is given by: $x = \frac{1}{3}(2t+3)^{3/2}; y = \frac{t^2}{2} + t$. Find the distance it travel between $t = 0$ and $t = 3$.

Solution:

$$x = \frac{1}{3}(2t+3)^{\frac{3}{2}}, \quad y = \frac{t^2}{2} + t \quad t = 0, t = 3$$

$$x = \frac{1}{3}(2t+3)^{\frac{3}{2}} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{1}{3} \times \frac{3}{2}(2t+3)^{\frac{1}{2}} \times 2 = \sqrt{2t+3}$$

$$y=\frac{t^2}{2}+t \quad \Rightarrow \quad \frac{dy}{dt}=t+1$$

$$l=\int_0^3\sqrt{(2t+3)+(t+1)^2}\ dt$$

$$=\int_0^3\sqrt{(2t+3)+(t^2+2t+1)}\ dt$$

$$\int_0^3\sqrt{2t+3+t^2+2t+1}\ dt=\sqrt{t^2+4t+4}\ dt$$

$$\int_0^3\sqrt{(t+2)^2}\ dt=\int_0^3(t+2)\,dt$$

$$=\frac{t^2}{2}+2t\Big|_0^3=\frac{9}{2}+6-0=\frac{21}{2}.$$

(Ans. $\frac{21}{2}$)

PROBLEM 7.8

Find the area of the surface generated by rotating about the x-axis the arc of the curve $y=x^3$ between $x=0$ and $x=1$

Solution:

$$y=x^3 \quad x=0 \quad \text{and} \quad x=1$$

$$y=x^3 \quad \Rightarrow \quad \frac{dy}{dx}=3x^2$$

$$s=\int_a^b 2\pi y \quad \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$=2\pi\int_0^1 x^3 \quad \sqrt{1+\left(3x^2\right)^2}\ dx$$

$$=2\pi\int_0^1 x^3\sqrt{1+9x^4}\ dx \quad \Rightarrow \quad \frac{2\pi}{36}\int_0^1 36x^3 \quad (1+9x^4)^{\frac{1}{2}}\ dx$$

$$=\frac{\pi}{18}\int_0^1 36x^3 \quad (1+9x^4)^{\frac{1}{2}}\ dx$$

$$= \frac{\pi}{18}\left[\frac{(1+9x^4)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$$

$$= \frac{\pi}{27}(10\sqrt{10}-1).$$

(Ans. $(\frac{\pi}{27}(10\sqrt{10}-1))$)

PROBLEM 7.9

Find the area of the surface generated by rotating about the y-axis the arc of the curve $y = x^2$ between $(0, 0)$ and $(2, 4)$.

Solution:

$$y = x^2 \quad (0,0) \quad (2,4) \quad \text{rotating about } y\text{-axis.}$$

$$x^2 = y$$

$$\therefore \quad x = \mp\sqrt{y} \xrightarrow{\text{since}} x = \sqrt{y}$$

$$0 \le x \le 2$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$$\therefore \quad s = \int_c^d 2\pi\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= \int_0^4 2\pi\sqrt{y}\sqrt{1+\frac{1}{4y}}\,dy = 2\pi\int_0^4 \sqrt{y}\left(1+\frac{1}{4y}\right)^{\frac{1}{2}} dy$$

$$= 2\pi\int_0^4 \sqrt{y}\,\frac{\sqrt{4y+1}}{2\sqrt{y}}\,dy \Rightarrow \pi\int_0^4 (4y+1)^{\frac{1}{2}}\,dy$$

$$= \frac{\pi}{4}\int_0^4 (4y+1)^{\frac{1}{2}}\,dy$$

$$= \frac{\pi}{4}\left[\frac{(4y+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$\Rightarrow \quad = \frac{\pi}{6}\left(17\sqrt{17}-1\right)$$

(Ans. $\frac{\pi}{6}(17\sqrt{17}-1)$)

PROBLEM 7.10

Find the area of the surface generated by rotating about the y-axis the curve

$$y = \frac{x^2}{2} + \frac{1}{2};\ 0 \le x \le 1.$$

Solution:

$$y = \frac{x^2}{2} + \frac{1}{2} \qquad 0 \le x \le 1$$

$$y = \frac{x^2+1}{2} \Rightarrow 2y = x^2 + 1 \Rightarrow 2y - 1 = x^2$$

$$\therefore \quad x = \mp\sqrt{2y-1} \xrightarrow{\text{since}} 0 \le x \le 1 \quad \therefore \quad x = \sqrt{2y-1}$$

$$\therefore \quad \frac{dx}{dy} = \frac{1}{2}(2y-1)^{\frac{-1}{2}} \times 2 = \frac{1}{\sqrt{2y-1}}$$

$$\left.\begin{array}{l} \text{at } x = 0 \Rightarrow y = \frac{1}{2} \\ \text{at } x = 1 \Rightarrow y = 1 \end{array}\right\} \Rightarrow \frac{x^2}{2} + \frac{1}{2}$$

$$\therefore \quad S = \int_{\frac{1}{2}}^{1} 2\pi\sqrt{2y-1} \times \sqrt{1 + \frac{1}{2y-1}}\, dy$$

$$= 2\pi \int_{\frac{1}{2}}^{1} \sqrt{2y-1} \times \sqrt{\frac{2y-1+1}{2y-1}}\, dy$$

$$= 2\pi \int_{\frac{1}{2}}^{1} \sqrt{2y}\, dy$$

$$= 2\sqrt{2}\pi \int_{\frac{1}{2}}^{1} (y)^{\frac{1}{2}}\, dy$$

$$= 2\sqrt{2}\pi \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_{\frac{1}{2}}^{1} = \frac{4\sqrt{2}}{3}\pi\left(1 - \frac{1}{2\sqrt{3}}\right)$$

$$= \frac{2}{3}\pi(2\sqrt{2} - 1)$$

(Ans. $\frac{2}{3}\pi(2\sqrt{2}-1)$)

PROBLEM 7.11

The curve described by the particle $P(x, y)\ x = t + 1, y = \frac{t^2}{2} + t$ from $t = 0$ to $t = 4$ is rotated about the y-axis. Find the surface area that is generated.

Solution:

$P(x, y), \quad x = t + 1, \qquad y = \frac{t^2}{2} + t$

From $\qquad t = 0 \quad to \quad t = 4$

$$x = t + 1 \quad \Rightarrow \quad \frac{dx}{dt} = 1$$

$$y = \frac{t^2}{2} + t \quad \Rightarrow \quad \frac{dy}{dt} = t + 1$$

$$S = 2\pi \int_0^4 x \sqrt{1 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_0^4 (t+1)\sqrt{(1 + (t+1)^2}\, dt$$

$$= 2\pi \int_0^4 (t+1)\ (1 + (t+1)^2)^{\frac{1}{2}}$$

$$= 2\pi \int_0^4 2(t+1)\ (1 + (t+1)^2)^{\frac{1}{2}}\, dt$$

$$= \pi \left[\frac{(1 + (t+1)^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 = \frac{2}{3}\pi\,(26\sqrt{26} - 2\sqrt{2})$$

$$= \frac{2\sqrt{2}}{3}\pi\left(13\sqrt{13} - 1\right)$$

(Ans. $\frac{2\sqrt{2}}{3}\pi(13\sqrt{13} - 1)$)

CHAPTER 8

MATRICES AND DETERMINANTS

PROBLEM

PROBLEM 8.1

Let $\boldsymbol{A}=\begin{bmatrix}1 & 2\\0 & 4\end{bmatrix}, \boldsymbol{B}=\begin{bmatrix}1 & 2 & 3\\-1 & 4 & -2\end{bmatrix}, \boldsymbol{C}=\begin{bmatrix}3 & 1\\4 & -1\\0 & 2\end{bmatrix}, \boldsymbol{D}=\begin{bmatrix}1 & 0 & 4\\0 & 1 & 2\\0 & -1 & 1\end{bmatrix}$, **and** $\boldsymbol{E}=\begin{bmatrix}3 & -1\\4 & 2\end{bmatrix}$. **Find**

(*a*) $\boldsymbol{AB}$ (*b*) $\boldsymbol{DC}$ (*c*) $(\boldsymbol{D}+\boldsymbol{I})\boldsymbol{C}$ (*d*) $\boldsymbol{DC}+\boldsymbol{C}$ (*e*) $\boldsymbol{DCB}$
(*f*) $\boldsymbol{EI}$ (*g*) $3\boldsymbol{A}+\boldsymbol{E}$ (*h*) $-5\boldsymbol{E}+\boldsymbol{A}$ (*i*) $\boldsymbol{E}(2\boldsymbol{B})$

Solution:

Given $A=\begin{bmatrix}1 & 2\\0 & 4\end{bmatrix},\quad B=\begin{bmatrix}1 & 2 & 3\\-1 & 4 & -2\end{bmatrix},\quad C=\begin{bmatrix}3 & 1\\4 & -1\\0 & 2\end{bmatrix}$

$$D=\begin{bmatrix}1 & 0 & 4\\0 & 1 & 2\\0 & -1 & 1\end{bmatrix},\quad E=\begin{bmatrix}3 & -1\\4 & 2\end{bmatrix}$$

$$(a) \qquad AB = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 1 + 2\times(-1) & 1\times 2 + 2\times 4 & 1\times 3 + 2\times(-2) \\ 0\times 1 + 4\times(-1) & 0\times 2 + 4\times 4 & 0\times 3 + 4\times(-2) \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 10 & -1 \\ -4 & 16 & -8 \end{bmatrix} \qquad \left(\textbf{Ans. } \begin{bmatrix} \mathbf{-1} & \mathbf{10} & \mathbf{-1} \\ \mathbf{-4} & \mathbf{16} & \mathbf{-8} \end{bmatrix}\right)$$

$$(b) \qquad DC = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 4 & -1 \\ 0 & 2 \end{bmatrix}$$

$$BC = \begin{bmatrix} 1\times 3 + 0\times 4 + 4\times 0 & 1\times 1 + 0\times(-1) + 4\times 2 \\ 0\times 3 + 1\times 4 + 2\times 0 & 0\times 1 + 1\times(-1) + 2\times 2 \\ 0\times 3 + (-1)\times 4 + 1\times 0 & 0\times 1 + (-1)\times 1\times 2 \end{bmatrix}$$

$$BC = \begin{bmatrix} 3 & 9 \\ 4 & 3 \\ -4 & 3 \end{bmatrix} \qquad \left(\textbf{Ans. } \begin{bmatrix} \mathbf{3} & \mathbf{9} \\ \mathbf{4} & \mathbf{3} \\ \mathbf{-4} & \mathbf{3} \end{bmatrix}\right)$$

$$(c) \ (D+I)\,C = \left[\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] \begin{bmatrix} 3 & 1 \\ 4 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 4 \\ 0 & 2 & 2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 3 + 0\times 4 + 4\times 0 & 2\times 1 + 0\times(-1) + 4\times 2 \\ 0\times 3 + 2\times 4 + 2\times 0 & 0\times 1 + 2\times(-1) + 2\times 2 \\ 0\times 3 + (-1)\times 4 + 2\times 0 & 0\times 1 + (-1)\times(-1) + 2\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 10 \\ 8 & 2 \\ -4 & 5 \end{bmatrix} \qquad \left(\textbf{Ans. } \begin{bmatrix} \mathbf{6} & \mathbf{10} \\ \mathbf{8} & \mathbf{2} \\ \mathbf{-4} & \mathbf{5} \end{bmatrix}\right)$$

(*d*) $DC+C=\begin{bmatrix}1&0&4\\0&1&2\\0&-1&1\end{bmatrix}\begin{bmatrix}3&1\\4&-1\\0&2\end{bmatrix}+\begin{bmatrix}3&1\\4&-1\\0&2\end{bmatrix}$

$$DC=\begin{bmatrix}1\times3+0\times4+4\times0 & 1\times1+0\times(-1)+4\times2\\0\times3+1\times4+2\times0 & 0\times1+1\times(-1)+2\times2\\0\times3+(-1)\times4+1\times0 & 0\times1+(-1)\times(-1)+1\times2\end{bmatrix}+\begin{bmatrix}3&1\\4&-1\\0&2\end{bmatrix}$$

$$=\begin{bmatrix}3&9\\4&3\\-4&3\end{bmatrix}+\begin{bmatrix}3&1\\4&-1\\0&2\end{bmatrix}=\begin{bmatrix}6&10\\8&2\\-4&5\end{bmatrix}$$

(Ans. $\begin{bmatrix}\mathbf{6}&\mathbf{10}\\\mathbf{8}&\mathbf{2}\\\mathbf{-4}&\mathbf{5}\end{bmatrix}$**)**

(*e*)

$$DCB=\begin{bmatrix}1&0&4\\0&1&2\\0&-1&1\end{bmatrix}\begin{bmatrix}3&1\\4&-1\\0&2\end{bmatrix}\begin{bmatrix}1&2&3\\-1&4&-2\end{bmatrix}$$

First we find $DC=\begin{bmatrix}1&0&4\\0&1&2\\0&-1&1\end{bmatrix}\begin{bmatrix}3&1\\4&-1\\0&2\end{bmatrix}$

$$=\begin{bmatrix}1\times3+0\times4+4\times0 & 1\times1+0\times(-1)+4\times2\\0\times3+1\times4+2\times0 & 0\times1+1\times(-1)+2\times2\\0\times3+(-1)\times4+1\times0 & 0\times1+(-1)\times(-1)+1\times2\end{bmatrix}$$

$$=\begin{bmatrix}3&9\\4&3\\-4&3\end{bmatrix}$$

$\therefore$ $$DCB=\begin{bmatrix}3&9\\4&3\\-4&3\end{bmatrix}\begin{bmatrix}1&2&3\\-1&4&-2\end{bmatrix}$$

$$BCB=\begin{bmatrix}3\times1+9\times(-1) & 3\times2+9\times4 & 3\times3+9\times(-2)\\4\times1+3\times(-1) & 4\times2+3\times4 & 4\times3+3\times(-2)\\-4\times1+3\times(-1) & -4\times2+3\times4 & -4\times3+3\times(-2)\end{bmatrix}$$

$$= \begin{bmatrix} -6 & 42 & -9 \\ 1 & 20 & 6 \\ -7 & 4 & -18 \end{bmatrix} \qquad \left(\textbf{Ans. } \begin{bmatrix} \mathbf{-6} & \mathbf{42} & \mathbf{-9} \\ \mathbf{1} & \mathbf{20} & \mathbf{6} \\ \mathbf{-7} & \mathbf{4} & \mathbf{-18} \end{bmatrix}\right)$$

(*f*) $$EI = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 1 + (-1)\times 0 & 3\times 0 + (-1)\times (1) \\ 3\times 1 + 2\times 0 & 4\times 0 + 2\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix} \qquad \left(\textbf{Ans. } \begin{bmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{4} & \mathbf{2} \end{bmatrix}\right)$$

(*g*) $$3A + E = 3\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 \\ 0 & 12 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 4 & 14 \end{bmatrix} \qquad \left(\textbf{Ans. } \begin{bmatrix} \mathbf{6} & \mathbf{5} \\ \mathbf{4} & \mathbf{14} \end{bmatrix}\right)$$

(*h*) $$-5E + A = -5\begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 5 \\ -20 & -10 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -14 & 7 \\ -20 & -6 \end{bmatrix} \qquad \left(\textbf{Ans. } \begin{bmatrix} \mathbf{-14} & \mathbf{7} \\ \mathbf{-20} & \mathbf{-6} \end{bmatrix}\right)$$

(*i*) $$E(2B) = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}\left[2\begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & -2 \end{bmatrix}\right] \qquad \left(\textbf{Ans. } \begin{bmatrix} \mathbf{8} & \mathbf{4} & \mathbf{22} \\ \mathbf{3} & \mathbf{32} & \mathbf{16} \end{bmatrix}\right)$$

$$= \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 & 6 \\ -2 & 8 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 2 + (-1)\times -2 & 3\times 4 + (-1)\times 8 & 3\times 6 + (-1)\times (-4) \\ 4\times 2 + 2\times (-2) & 4\times 4 + 2\times 8 & 4\times 6 + 2\times (-4) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 & 22 \\ 3 & 32 & 16 \end{bmatrix}$$

PROBLEM 8.2

Find the value of x if

$$\begin{bmatrix} x & 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ -7 \\ 5/4 \end{bmatrix} = 0$$

Solution:

$$\text{Find the value of } x\begin{bmatrix} x & 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix}.$$

$$\Rightarrow \quad [x\times(2)+4(1)+1\times 0 \quad x\times(1)+4\times(0)+1\times 2x\times(0)+4\times 2+1\times 4]$$

$$[2x+4x+2 \ \ 12]\times\begin{bmatrix} x \\ -7 \\ \frac{5}{4} \end{bmatrix} = 0$$

$$= (2x+4)\times(x)+(x+2)\times(-7)+12\times 5/(4) = 0$$

$$2x^2+4x-7x-14+15 = 0$$

$$= 2x^2-3x+1 = 0$$

$$(2x-1)(x-1) = 0$$

$$\text{Either } (2x-1) = 0$$

$$x = \frac{1}{2}$$

OR

$$(x-1) = 0$$

$$x = 1$$

(Ans. $x = \frac{1}{2}$ or $x = 0$)

PROBLEM 8.3

Find v and w if $[5 \quad w] = v[-2 \quad 1]$

Solution:

$$\begin{bmatrix}5 & w\end{bmatrix} = v[-2 \quad 1] \Rightarrow [5 \quad w] = \begin{bmatrix}-2v & v\end{bmatrix}$$

$$-2v = 5 \quad \Rightarrow \quad v = -\frac{5}{2}$$

$$w = v \quad \Rightarrow \quad w = -\frac{5}{2}$$

PROBLEM 8.4

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix}$. **Find**

(a) $2A + B'$ **(b)** $B'A' - 1$

Solution:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \Rightarrow \quad 2A = \begin{bmatrix} 2 & -2 & 4 \\ 0 & 2 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix} \quad \Rightarrow \quad B' = \begin{bmatrix} 0 & -1 & 5 \\ 2 & 3 & -2 \end{bmatrix}$$

$$\therefore \qquad 2A + B' \quad \Rightarrow \begin{bmatrix} 2 & -2 & 4 \\ 0 & 2 & 8 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 5 \\ 2 & 3 & -2 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} -2 & -3 & 9 \\ 2 & 5 & 6 \end{bmatrix}$$

$$\left(\textbf{Ans. } (a) \begin{bmatrix} 2 & -3 & 9 \\ 2 & 5 & 6 \end{bmatrix} (b) \begin{bmatrix} 10 & 19 \\ -5 & -6 \end{bmatrix} \right)$$

PROBLEM 8.5

Let $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & 2 \end{bmatrix}$. **Find** $(2A - I)B'$ **and show that** $(BA)' = A'B'$.

Solution:

(*a*) Find $(2A-I)B$ $\qquad I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ Note

$$\therefore \qquad (2A-I)\Rightarrow\begin{bmatrix}6&0&2\\0&2&4\\-2&2&10\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}5&0&2\\0&1&4\\-2&2&9\end{bmatrix}$$

$$B=\begin{bmatrix}1&-1\\-2&3\\0&2\end{bmatrix}\text{as } B=\begin{bmatrix}1&-2&0\\-1&3&2\end{bmatrix}$$

$$\therefore \qquad (2A-I)B'=\begin{bmatrix}5&0&2\\0&1&4\\-2&2&9\end{bmatrix}\begin{bmatrix}1&-1\\-2&3\\0&2\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}5\times(1)+0\times(-2)+2\times0 & 5\times(-1)+0\times3+2\times2\\0\times(1)+1\times(-2)+4\times(0) & 0\times(-1)+(-1)\times(3)+4\times(2)\\-2\times(1)+2\times(-1)+9\times(0) & -2\times(-1)+2\times3+9\times2\end{bmatrix}$$

$$=\begin{bmatrix}5&-1\\-2&11\\-6&26\end{bmatrix}$$

Now, $$BA=\left[\begin{bmatrix}1&-2&0\\-1&3&2\end{bmatrix}\begin{bmatrix}3&0&1\\0&1&2\\-1&1&5\end{bmatrix}\right]$$

$$=\begin{bmatrix}3&-2&-3\\-5&5&5\end{bmatrix}\quad\therefore\quad (BA)^1=\begin{bmatrix}3&-5\\-2&5\\-3&5\end{bmatrix}$$

and $$A'B' = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -2 & 5 \\ -3 & 5 \end{bmatrix} = \text{L.H.S.}$$

Hence, $$(BA)' = A'B'$$

PROBLEM 8.6

For what value of x will $\begin{vmatrix} x & x & 1 \\ 2 & 0 & 5 \\ 6 & 7 & 1 \end{vmatrix} = 0$?

Solution: Expand the determinant

$$\therefore \quad x\begin{vmatrix} 0 & 5 \\ 7 & 1 \end{vmatrix} - x\begin{vmatrix} 2 & 5 \\ 6 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 \\ 6 & 7 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(x-35) - x(2-30) + 1(14-0) = 0$$

$$\Rightarrow \quad x(-35) - x(2-30) + 1(14-0) = 0$$

$$\Rightarrow \quad -35x + 28x + 14 = 0$$

$$\Rightarrow \quad -7x = -14$$

$$\Rightarrow \quad x = 2$$

(Ans. $x = 2$)

PROBLEM 8.7

Let A be an arbitrary 3 by 3 matrix, and let R_{12} be the matrix obtained from the 3 by 3 identity matrix by interchanging rows 1 and 2: $R_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (*a*) Compute $R_{12}A$ and show that you would get the same result by interchanging rows 1 and 2 of A. (*b*) Compute AR_{12} and show that the result is what you obtain by interchanging columns 1 and 2 of A.

Solution:

$$A = \begin{bmatrix} a_{11} & a_{22} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$R_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(a)\ R_{12}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{21} & a_{22} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ interchanging columns 1 and 2 of } A$$

$$(b)\ AR_{12} = \begin{bmatrix} a_{11} & a_{22} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix} \text{interchanging columns 1 and 2 of } A.$$

$$\left(\textbf{Ans. } (a) \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} (b) \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix} \right)$$

PROBLEM 8.8

Solve the following determinants:

$$(a) \begin{vmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 2 \end{vmatrix} \quad (b) \begin{vmatrix} 2 & -1 & -2 \\ -1 & 2 & 1 \\ 3 & 0 & -3 \end{vmatrix} \quad (c) \begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

$$(d) \begin{vmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{vmatrix} \quad (e) \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & -1 & 0 & 7 \\ 3 & 0 & 2 & 1 \end{vmatrix} \quad (f) \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 2 \end{vmatrix}$$

$$(\boldsymbol{g})\ \begin{vmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{-} & \mathbf{2} & \mathbf{6} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{3} \\ \mathbf{-2} & \mathbf{2} & \mathbf{0} & \mathbf{-5} \end{vmatrix} \qquad (\boldsymbol{h})\ \begin{vmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{vmatrix}$$

Solution:

$$(a)\qquad \begin{vmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 2 \end{vmatrix} = 2\begin{vmatrix} 5 & 2 \\ 2 & 2 \end{vmatrix} - 3\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix}$$

$$|A| = 12 - 18 + 3$$

$$= -3$$

$$(b)\qquad \begin{vmatrix} 2 & -1 & -2 \\ -1 & 2 & -1 \\ 3 & 0 & -3 \end{vmatrix} = -12 + 0 + 12$$

$$= 0$$

$$(c)\qquad \begin{vmatrix} 2 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 2 & 1 \end{vmatrix} = 0 + 0 + 4 - (0 + 12 - 1)$$

$$= 4 - (11) = -7$$

$$(d)\qquad \begin{vmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{vmatrix} = (2 + 0 + 0) - (-4 + 0 + 0)$$

$$2 + 4 = 6$$

$$(e)\ \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & -1 & 0 & 7 \\ 3 & 0 & 2 & 1 \end{vmatrix} \Rightarrow -1\begin{vmatrix} 0 & -2 & 1 \\ -1 & 0 & 7 \\ 0 & 2 & 1 \end{vmatrix} - \begin{vmatrix} 0 & -2 & 1 \\ 0 & 0 & 7 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= +1\begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} - 1\left[3\begin{vmatrix} -2 & 1 \\ 0 & 7 \end{vmatrix}\right]$$

$$= -2 - 2 - 3\,(-14)$$

$$= 2 - 2 + 42 = 38$$

(*f*) $$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 3 & 2 \end{vmatrix}$$

$$\Rightarrow \quad (1)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 3 & 2 \end{vmatrix}$$

Expend this along the first column:

$$1\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 = 1$$

(*g*) $$\begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & 1 & 2 & 6 \\ 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & -5 \end{vmatrix}$$

$$(1)\begin{vmatrix} 1 & 2 & 6 \\ 0 & 2 & 3 \\ 2 & 0 & -5 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 2 & 6 \\ 1 & 2 & 3 \\ -2 & 0 & -5 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 & 6 \\ +1 & 0 & 3 \\ -2 & 2 & -5 \end{vmatrix} (-3)\begin{vmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ -2 & 2 & 0 \end{vmatrix}$$

$$(1)[(-10+12+0)-(24+0+0)]+(1)(-20-12)-24-10]$$
$$(+2)[0+(-6)+(+12)-(0+12-5)](-3)[(0-4+4)-(0+8+0)]$$
$$=(-22)+(2)+(-2)+24=2$$

(*h*) $$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{vmatrix} = 0 - \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1$$

(*Ans.* (*a*) – 5 (*b*) 0 (*c*) –7 (*d*) 6
(*e*) 38 (*f*) 1 (*g*) 2 (*h*) –1)

PROBLEM 8.9

Solve the following systems of equations:

(*a*) $x + 8y = 4$
$3x - y = -13$

(*b*) $2x + 3y = 5$
$3x - y = 2$

(*c*) $x + y + z = 2$
$2x - y + z = 0$
$x + 2y - z = 4$

(*d*) $2x + y - z = 2$
$x - y + z = 7$
$2x + 2y + z = 4$

(*e*) $2x - 4y = 6$
$x + y + z = 1$
$5y + 7z = 10$

(*f*) $x - z = 3$
$2y - 2z = 2$
$2x + z = 3$

(*g*) $x_1 + x_2 - x_3 + x_4 = 2$
$x_1 - x_2 + x_3 + x_4 = -1$
$x_1 + x_2 + x_3 - x_4 = 2$
$x_1 + x_3 + x_4 = -1$

(*h*) $2x - 3y + 4z = -19$
$6x + 4y - 2z = 8$
$x + 5y + 4z = 23$

Solution:

(*a*)
$$x + 8y = 4$$
$$3x - y = -13$$
$$|A| = \begin{vmatrix} 1 & 8 \\ 3 & -1 \end{vmatrix} = -1 - 24$$
$$= -25$$
$$|A_1| = \begin{vmatrix} 4 & 8 \\ -13 & -1 \end{vmatrix} = -4 + 104$$
$$= 100$$
and
$$|A_2| = \begin{vmatrix} 1 & 4 \\ 3 & -13 \end{vmatrix} = -13 - 12$$
$$= -25$$
$\therefore$
$$x = \frac{|A_1|}{|A|} = \frac{100}{-25} = -4$$
$$y = \frac{|A_2|}{|A|} = \frac{-25}{-25} = -1$$

(Ans. $x = -4, y = -1$)

(b) $2x+3y=5$

$3x-y=2$

$$|A|=\begin{vmatrix}2 & 3\\3 & -1\end{vmatrix}$$

$\Rightarrow$ $-2-9=-11$

$$|A_1|=\begin{vmatrix}5 & 3\\2 & -1\end{vmatrix}$$

$$=-5-6=-11$$

and $$|A_2|=\begin{vmatrix}2 & 5\\3 & 2\end{vmatrix}$$

$4-15=-11$

$\therefore$ $$x=\frac{|A_1|}{A}=\frac{-11}{-11}=1$$

and $$y=\frac{|A_2|}{A}=\frac{-11}{-11}=1$$

$y=1$ **(Ans. $x=1, y=1$)**

(c) Using Crammer's rule,

$x+y+z=2$

$2x-y+z=0$

$x+2y-z=4$

$$|A|=\begin{vmatrix}1 & 1 & 1\\2 & -1 & 1\\1 & 2 & -1\end{vmatrix}$$

$$-(-1+2-2)=(1+1+4)$$

$$=6-(-1)$$

$$=7$$

$\therefore$ $$|A_1|=\begin{vmatrix}2 & 1 & 1\\2 & -1 & 1\\1 & 2 & -1\end{vmatrix}$$

$$=(2+4+0)-(-4+4+0)$$

$$|A_1|=6$$

$$\therefore \qquad |A_2| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 4 & -1 \end{vmatrix}$$

$$= (0 + 2 + 8) - (0 + 4 - 4)$$

$$\therefore \qquad |A_2| = 10$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 0 \\ 1 & 2 & 4 \end{vmatrix} \begin{matrix} 1 & 1 \\ 2 & -1 \\ 1 & 2 \end{matrix}$$

$$(-4 + 0 + 8) - (-2 + 0 + 8)|A_3|$$

$$\therefore \qquad |A_3| = -2$$

$$x = \frac{|A_1|}{|A|} = \frac{6}{7}$$

$$y = \frac{|A_2|}{|A|} = \frac{10}{7}$$

$$z = \frac{|A_3|}{|A|} = \frac{-2}{7}$$

(Ans. $\boldsymbol{x = \frac{6}{7}, y = \frac{10}{7}, z = \frac{-2}{7}}$**)**

(*d*)

$$2x + y - z = 2$$

$$x - y + z = 7$$

$$2x + 2y + z = 4$$

$$|A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 1 \end{vmatrix}$$

$$= (-2 + 2 - 2) - (2 + 4 + 1)$$

$$|A| = -9$$

$$|A_1| = \begin{vmatrix} 2 & 1 & -1 \\ 7 & -1 & 1 \\ 4 & 2 & 1 \end{vmatrix}$$

$$= (-2 + 4 - 14) - (4 + 4 + 7)$$

$$|A_1| = -27$$

$$x = \frac{-27}{-9}$$

$$x = 3$$

$$|A_2| = \begin{vmatrix} 2 & 2 & -1 \\ 1 & 7 & 1 \\ 2 & 4 & 1 \end{vmatrix}$$

$$= (14 + 4 - 4) - (-14 + 8 + 2)$$

$$|A_2| = 18$$

$$y = \frac{18}{-9}$$

$$y = -2$$

$$|A_3| = \begin{vmatrix} 2 & 1 & 2 \\ 1 & -1 & 7 \\ 2 & 2 & 4 \end{vmatrix}$$

$$|A_3| = -18$$

$$z = \frac{-18}{-9} = 2$$

(Ans. $x = 3$, $y = -2$, $z = 2$)

(*e*) $2x - 4y = 6$

$x + y + z = 1$

$5y + 7z = 10$

$$|A| = \begin{vmatrix} 2 & -4 & 0 \\ 1 & 1 & 1 \\ 0 & 5 & 7 \end{vmatrix}$$

$$= (14 + 0 + 0) - (0 + 10 - 28)$$

$$|A| = 14 + 18 = 32$$

$$|A_1| = \begin{vmatrix} 6 & -4 & 0 \\ 1 & 1 & 1 \\ 10 & 5 & 7 \end{vmatrix}$$

$$= (42 - 40 + 0) - (0 + 30 - 28)$$

$$= 2 - 2 = 0$$

$$\therefore \qquad x = \frac{0}{32} = 0$$

$$|A_2| = \begin{vmatrix} 2 & 6 & 0 \\ 1 & 1 & 1 \\ 0 & 10 & 7 \end{vmatrix}$$

$$= (14 + 0 + 0) - (0 + 20 + 42)$$

$$= 14 - 62 = -48$$

$$\therefore \qquad y = \frac{-48}{32} = -1.5$$

$$|A_3| = \begin{vmatrix} 2 & -4 & 6 \\ 1 & 1 & 1 \\ 0 & 5 & 10 \end{vmatrix}$$

$$= (20 + 0 + 30) - (0 + 10 - 40)$$

$$|A_3| = 50 + 30 = 80$$

$$\therefore \qquad z = \frac{80}{32} = \frac{5}{2} = 2.5$$

(Ans. $x = 0, y = -1.5, z = 2.5$)

(*f*)

$$x - z = 3$$

$$2y - 2z = 2$$

$$2x + z = 3$$

$$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= (2 + 0 + 0) - (-4 + 0 + 0)$$

$$|A| = 2 + 4 = 6$$

$$|A_1| = \begin{vmatrix} 3 & 0 & -1 \\ 2 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= (6 + 0 + 0) - (-6 + 0 + 0)$$

$$|A_1| = 6 + 6 = 12$$

$$\therefore \qquad x = \frac{12}{6} = 2$$

$$|A_2| = \begin{vmatrix} 1 & 3 & -1 \\ 0 & 2 & -2 \\ 2 & 3 & 1 \end{vmatrix} \begin{matrix} 1 & 3 \\ 0 & 2 \\ 2 & 3 \end{matrix}$$

$$= (2 - 12 + 0) - (-4 + 6 + 0)$$

$$= -10 + 10 = 0$$

$\therefore \qquad y = 0$

$$|A_3| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 2 & 0 & 3 \end{vmatrix}$$

$$= (6 + 0 + 0) - (12 - 0 + 0)$$

$$= 6 - 12 = -6$$

$\therefore \qquad z = \dfrac{-6}{6} = -1$

(Ans. $\boldsymbol{x = 2, y = 0, z = -1}$)

(g)
$$x_1 + x_2 - x_3 + x_4 = 2$$
$$x_1 - x_2 + x_3 + x_4 = -1$$
$$x_1 + x_2 + x_3 - x_4 = 2$$
$$x_1 + x_3 + x_4 = -1$$

The augmented matrix $[A : B]$ is

$$\begin{matrix} -R_1 + R_2 \\ -R_1 + R_3 \\ \hline -R_1 + R_4 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 1 & : 2 \\ 1 & -1 & 1 & 1 & : -1 \\ 1 & 1 & 1 & -1 & : 2 \\ 1 & 0 & 1 & 1 & : -1 \end{bmatrix}$$

$$\frac{1}{2}R_2 + R_1$$

$$\begin{matrix} -\frac{1}{2}R_2 \\ \frac{1}{2}R_3 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 1 & : 2 \\ 0 & -2 & 1 & 0 & : -3 \\ 0 & 0 & 2 & -2 & : 0 \\ 0 & -1 & 2 & 0 & : -3 \end{bmatrix}$$

$$\frac{1}{2}R_2 + R_4$$

$$\begin{matrix} \\ R_3+R_2 \\ -R_3+R_4 \\ \\ \end{matrix}\begin{bmatrix} 1 & 0 & 0 & 1 & :\frac{1}{2} \\ 0 & 1 & -1 & 0 & :\frac{3}{2} \\ 0 & 0 & 1 & -1 & :0 \\ 0 & 0 & 1 & 0 & :\frac{-3}{2} \end{bmatrix}$$

$$\begin{matrix} -R_4+R_1 \\ R_4+R_2 \\ R_4+R_3 \end{matrix}\begin{bmatrix} 1 & 0 & 0 & 0 & :2 \\ 0 & 1 & 0 & 0 & :0 \\ 0 & 0 & 1 & 0 & :-\frac{3}{2} \\ 0 & 0 & 0 & 1 & :-\frac{3}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 1 & :\frac{1}{2} \\ 0 & 1 & 0 & -1 & :\frac{3}{2} \\ 0 & 0 & 1 & -1 & :0 \\ 0 & 0 & 0 & 1 & :-\frac{3}{2} \end{bmatrix}$$

$$\Rightarrow \qquad x_1 = 2, \quad x_2 = 0, \quad x_3 = -\frac{3}{2}, \quad x_4 = -\frac{3}{4}$$

(h)

$$2x - 3y + 4z = -19$$
$$6x + 4y - 2z = 8$$
$$x + 5y + 4z = 23$$

$$|A| = \begin{vmatrix} 2 & -3 & 4 \\ 6 & 4 & -2 \\ 1 & 5 & 4 \end{vmatrix}$$

$$= (32 + 6 + 120) - (16 - 20 - 72)$$

$$|A| = 234$$

$$|A_1| = \begin{vmatrix} -19 & -3 & 4 \\ 8 & 4 & -2 \\ 23 & 5 & 4 \end{vmatrix}$$

$$= (-304 + 138 + 160) - (368 + 190 - 96)$$

$$\therefore \qquad |A_1| = -468$$

$$|A_2| = \begin{vmatrix} 2 & -19 & 4 \\ 6 & 8 & -2 \\ 1 & 23 & 4 \end{vmatrix}$$

$$= (64 + 38 + 552) - (32 - 42 - 456)$$

$$\therefore \qquad |A_2| = 1170$$

$$|A_3| = \begin{vmatrix} 2 & -3 & -19 \\ 6 & 4 & 8 \\ 1 & 5 & 23 \end{vmatrix}$$

$$= (184 - 24 - 570) - (-76 + 80 - 414)$$

$$\therefore \qquad |A_3| = 0$$

$$x = \frac{|A_1|}{|A|} = \frac{-468}{234} = -2$$

$$y = \frac{|A_2|}{|A|} = \frac{1170}{234} = 5$$

$$z = \frac{|A_3|}{|A|} = \frac{0}{234} = 0$$

(Ans. $x = -2, y = 5, z = 0$)

CHAPTER 9

COMPLEX NUMBERS

PROBLEMS

PROBLEM 9.1

Find the values of

(a) $(2+3i)(4-2i)$ (b) $(2-i)(-2+3i)$ (c) $(-1-2i)(2+i)$

Solution: (a) $(2+3i)(4-2i)$

$$\Rightarrow \quad 8-4i+12i-6i^2$$

$$i^2=-1$$

$$8-4i+12i+6$$

$$14+8i$$

(Ans. $14+8i$)

(b) $(2-i)(-2+3i)$

$$\Rightarrow \quad -4+6i+2i-3i^2$$

$$-4+8i+3$$

$$=-1+8i$$

(Ans. $-1+8i$)

(c) $(-1-2i)(2+i)$

$$\Rightarrow \quad -2+i+4i-2i^2$$

$$-2+5i+2$$

$$=-5i$$

(Ans. $-5i$)

PROBLEM 9.2

Show that $\left(\dfrac{\mp 1 \pm i\sqrt{3}}{2}\right)^6 = 1$ for all combinations of signs.

Solution:

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

$$\left(\frac{+1+i\sqrt{3}}{2}\right)^6 = \frac{(1+i\sqrt{3})^6}{2^6} = \frac{1}{64}(1+i\sqrt{3})^6$$

$$r = \sqrt{1+3} = \sqrt{4} = 2$$

$$\theta = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$$

$$\therefore \qquad z^6 = \frac{1}{64}\left(2e^{i\frac{\pi}{3}}\right)^6 = \frac{1}{64}\left(2^6 e^{i\frac{6}{3}\pi}\right) = e^{i2\pi}$$

$$z^n = r^n(\cos\theta + i\sin\theta)$$

$$\therefore \qquad \frac{1}{64}\ 1+i\sqrt{3}\ ^6 = \frac{1}{64}\ .\ 64[\cos 2\pi + i\sin 2\pi] = 1+0 = 1.$$

PROBLEM 9.3

Solve the following equation for the real numbers x and y:

$$\mathbf{(3-2i)(x+iy) = 2(x-2iy)+2i-1}$$

Solution: $\quad (3-2i)(x+iy) = 2(x-2iy)+2i-1$

$$3x+3iy-2xi-2yi^2 = 2x-4iy+2i-1$$

$$3x+3iy-2xi+2y-2x+4iy = 2i-1$$

$$(x+2y)+(7y-2x)i = 2i-1$$

$$\therefore \qquad x+2y = -1 \qquad \ldots(1)$$

$$7y-2x = 2 \qquad \ldots(2)$$

From Equation (1)

$x+2y=-1 \quad \Rightarrow \quad x=-1-2y$ substitute this into Equation (2)

$$7y-2(-1-2y) = 2 \quad \Rightarrow \quad 7y+2+4y = 2$$

$$7y + 4y = 0 \quad \Rightarrow \quad 11y = 0$$

$\therefore$ $y = 0$ substitute this into Equation (1)

$$x + 0 = -1$$

$\therefore$ $x = -1$ **(Ans. $x = -1; y = 0$)**

PROBLEM 9.4

Show that $|\bar{z}| = |z|$.

Solution:

$$z = x + iy \quad \Rightarrow \quad |z| = \sqrt{x^2 + y^2}$$

$$\bar{z} = x - iy \quad \Rightarrow \quad |\bar{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|$$

PROBLEM 9.5

Let $Re(z)$ and $Im(z)$ denote, respectively, the real and imaginary parts of z, and show that

(a) $z + \bar{z} = 2Re$
(b) $z - \bar{z} = 2i\,\text{Im}(z)$
(c) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2Re(z_1\,\bar{z}_2)$

Solution:

(a) $z + \bar{z} = 2Re(z)$

$$z = x + iy \quad \Rightarrow \quad \bar{z} = x - iy$$

$$z + \bar{z} = (x + iy) + (x - iy) = 2x = 2Re(z)$$

(b) $z - \bar{z} = 2i\,\text{Im}(z)$

$$z - \bar{z} = (x + iy) - (x - iy) = x + iy - x + iy$$

$$= 2iy = 2i\,\text{Im}(z)$$

(c) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2Re(z_1\,\bar{z}_2)$

L.H.S. $\Rightarrow$ $|z_1 + z_2|^2 \Rightarrow |x_1 + iy_1 + x_2 + iy_2|^2$

$$\Rightarrow \left(\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}\right)^2$$

$$= (x_1 + x_2)^2 + (y_1 + y_2)^2$$

R.H.S. $\Rightarrow$ $|z_1|^2 + |z_2|^2 + 2Re(z_1\ \overline{z}_2)$

$= \left(\sqrt{x_1^2 + y_1^2}\right)^2 + \left(\sqrt{x_2^2 + y_2^2}\right)^2 + 2\ Re\,((x_1 + iy_1) + (x_2 - iy_2))$

$x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2(x_1\ x_2 + y_1\ y_2)$

$x_1^2 + 2x_1x_2 + x_2^2 + y_1^2 + 2y_1y_2 + y_2^2$

$= (x_1 + x_2)^2 + (y_1 + y_2)^2 = \text{L.H.S.}$

(Ans. On the line $y = -x$)

PROBLEM 9.6

Graph the points $z = x + iy$ that satisfy the given conditions:

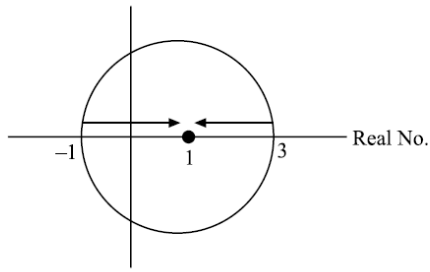
(*a*) $|z-1| = 2$ (*b*) $|z+1| = 1$ (*c*) $|z+i| = |z-1|$

Solution:

(*a*)

$$|z-1| = 2$$

$$\Rightarrow \quad |x + iy - 1| = 2$$



$$|(x-1) + iy| = 2$$

$$\Rightarrow \quad \sqrt{(x-1)^2 + y^2} = 2$$

$$(x-1)^2 + y^2 = 4$$

$\therefore$ $r = \text{radius} = 2$

The center of circle $(1, 0)$

(Ans. On the circle with center $(1, 0)$, radius 2)

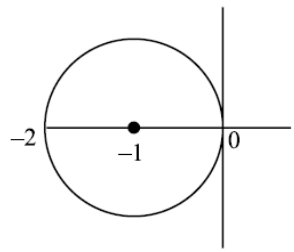
(*b*)

$$|z+1| = 1$$

$$|x + iy + 1| = 1$$

$$\Rightarrow \quad |(x+1) + iy| = 1$$

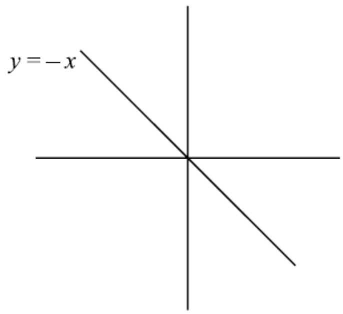
$$\sqrt{(x+1)^2+y^2}=1$$
$$(x-1)^2+y^2=1$$
$$r=1 \quad \text{center } (-1,0)$$



(Ans. On the circle with center $(-1, 0)$, radius 1)

(c) $$|z+i|=|z-1| \quad \Rightarrow \quad |x+iy+i|=|x+iy-1|$$
$$\sqrt{x^2+(y+1)^2}=\sqrt{(x-1)^2+y^2}$$
$$x^2+(y+1)^2=(x-1)^2+y^2$$
$$x^2+y^2+2y+1=x^2-2x+1+y^2$$
$$x^2+y^2+2y+1-x^2+2x-1-y^2=0$$
$$2x+2y=0 \quad \Rightarrow \quad x+y=0$$
$\therefore$ $$y=-x$$

x	$y=-x$
0	0
1	–1
2	–2
3	–3
–1	1
–2	2
–3	3



(Ans. On the line $y=-x$)

PROBLEM 9.7

Express the following complex number in exponential form with $r \geq 0$ and $-\pi < \theta < \pi$.

(a) $\left(1+\sqrt{-3}\right)^2$ **(b)** $\dfrac{1+i}{1-i}$ **(c)** $\dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}$ **(d)** $(2+3i)(1-2i)$

Solution:

(a) $\left(1+\sqrt{-3}\right)^2 \qquad i^2 = -1$

$$\left(1+\sqrt{-3}\right)^2 \Rightarrow \left(1+i\sqrt{3}\right)^2$$

$$r = \sqrt{1+3} \quad \Rightarrow \quad r = \sqrt{4} \Rightarrow r = 2$$

$$\theta = \tan^{-1}\frac{y}{x} \quad \Rightarrow \quad \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$$

$$\left(1+\sqrt{-3}\right)^2 = \left(2e^{\frac{\pi}{3}i}\right)^2 = 4e^{\frac{2\pi}{3}i} \qquad \left(\textbf{Ans. } 4e^{i\frac{2}{3}\pi}\right)$$

(b) $\dfrac{1+i}{1-i}$

$$\frac{1+i}{1-i}\times\frac{1+i}{1+i} = \frac{1+2i+i^2}{1-i^2} = \frac{1+2i-1}{1-i^2} = \frac{2i}{2} = i$$

$$r = \sqrt{0+(1)^2} = 1 \quad \theta = \tan^{-1}\frac{1}{0} = \frac{\pi}{2}$$

$\therefore$
$$r = \frac{1+i}{1-i} = e^{\frac{\pi}{2}i} \qquad \left(\textbf{Ans. } e^{i\frac{\pi}{2}}\right)$$

(c)
$$\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\times\frac{1+i\sqrt{3}}{1+i\sqrt{3}} = \frac{1+2\sqrt{3}i+3i^2}{1-3i^2} = \frac{1+2\sqrt{3}i-3}{1-3i^2}$$

$\Rightarrow$
$$\frac{-2+2\sqrt{3}i}{1+3} = \frac{-2+2\sqrt{3}i}{4} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

$\therefore$
$$r = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{2}{3}\pi$$

$\therefore$
$$\frac{1+i\sqrt{3}}{1-i\sqrt{3}} = e^{i\frac{2}{3}\pi} \qquad \left(\textbf{Ans. } e^{i\frac{2}{3}\pi}\right)$$

(d) $(2+3i)(1-2i) \Rightarrow \quad 2-4i+3i-6i^2$

$2-4i+3i+6 \Rightarrow \quad 8-i$

$\Rightarrow \qquad r = \sqrt{(8)^2 + (1)^2}$

$\sqrt{64+1} = \sqrt{65}$

$\theta = \tan^{-1}\left(\frac{-1}{8}\right) = \tan^{-1}(0.125)$

$\therefore \qquad z = re^{i\theta} \quad \Rightarrow \quad z = \sqrt{65} \quad e^{i\tan^{-1}(-0.125)}$

$\left(\textbf{Ans.} \sqrt{65}e^{i\tan^{-1}(0-1.25)}\right)$

PROBLEM 9.8

Find the three cube roots of 1.

Solution:

$$\sqrt[3]{1} \Rightarrow r \Rightarrow \sqrt{1+0} = 1 \quad \theta = \tan^{-1}\frac{0}{1} = 0$$

$\because \qquad z = \sqrt[n]{r}\ e^{i\left(\frac{\theta}{n}+k\frac{2\pi}{n}\right)}$

$\because$ Three roots $\qquad k = 0, 1, 2$

At $\qquad k = 0 \quad \Rightarrow \quad$ 1st root $w_0 = e^0 = 1 \quad z = 1$

at $\qquad k = 1 \quad \Rightarrow \quad$ 2nd root $w_1 = e^{i\left(\frac{0}{3}+1\times\frac{2\pi}{3}\right)} = e^{\frac{2\pi}{3}i}$

$\therefore \qquad z = \sqrt[n]{r}\,(\cos\theta + i\sin\theta)$

$$z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

at $\qquad k = 2 \quad \Rightarrow \quad$ 3rd root $w_2 = e^{i\left(\frac{0}{3}+2\times\frac{2\pi}{3}\right)} = e^{\frac{4}{3}\pi i}$

$\therefore \qquad z = \cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

PROBLEM 9.9

Find the two square roots of i.

Solution:

$$\sqrt{i} \Rightarrow r = \sqrt{0+1} = 1 \quad \text{and}$$

$$\theta = \tan^{-1}\frac{1}{0} = \frac{\pi}{2}$$

at $$k = 0 \Rightarrow \text{1st root } w_0 = e^{i\left(\frac{\pi/2}{2}+0\times\frac{2\pi}{2}\right)} = e^{\frac{\pi}{4}i}$$

$\therefore$ $$\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

at $$k = 1 \quad \Rightarrow \quad \text{2nd root } w_1 = e^{i\left(\frac{\pi/2}{2}+1\times\frac{2\pi}{2}\right)}$$

$$= e^{i\left(\frac{1}{2}\cdot\frac{\pi}{2}+\frac{2\pi}{2}\right)} = e^{i\left(\frac{\pi}{4}+\pi\right)} = e^{i\left(\frac{5}{4}\pi\right)}$$

$\therefore$ $$\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

$$\left(\textbf{Ans.} -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)$$

PROBLEM 9.10

Find the three cube roots of $(-8i)$.

Solution:

$$\sqrt[3]{-8i} \Rightarrow r = \sqrt{0+(-8)^2} = 8 \text{ and } \theta = \tan^{-1}\frac{-8}{0} = -\frac{\pi}{2}$$

at $$k = 0 \quad \Rightarrow \quad \text{1st root } w_0 = \sqrt[3]{8}\ e^{i\left(\frac{-\pi/2}{3}+0\times\frac{2\pi}{2}\right)} = \sqrt[3]{8}\ e^{-\frac{\pi}{6}i}$$

$$= 2e^{-\frac{\pi}{6}i}$$

$$\sqrt[3]{8} = 2.$$

$\therefore$ $$z = 2\left(\cos-\frac{\pi}{6} + i\sin-\frac{\pi}{6}\right) = \sqrt{3} - i$$

at $$k = 1 \quad \Rightarrow \quad \text{2nd root } w_1 = \sqrt[3]{8}\ e^{i\left(\frac{-\pi/2}{3}+\frac{2\pi}{3}\right)}$$

$$w_1 = 2\ e^{i\left(-\frac{\pi}{6}+\frac{2\pi}{3}\right)} = 2\ e^{i\left(\frac{-\pi+4}{6}\right)} = 2\ e^{i\frac{\pi}{2}}$$

$\therefore \qquad 2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=2i$

at $\qquad k=2 \quad \Rightarrow \quad \text{3rd root } w_2 = 2\, e^{i\left(\frac{-\pi/2}{3}+2\times\frac{2\pi}{3}\right)} = 2\, e^{i\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)}$

$= 2\, e^{i\left(\frac{-\pi+8\pi}{6}\right)} = 2\, e^{i\frac{7}{6}\pi}$

$\Rightarrow \qquad 2\left(\cos\frac{7}{6}\pi+i\sin\frac{7}{6}\pi\right)$

$= -\sqrt{3}-i.$ $\qquad$ $\left(\textbf{Ans.} -2i; \pm\sqrt{3}-i\right)$

PROBLEM 9.11

Find the six sixth roots of (64).

Solution:

$$\sqrt[6]{64} \Rightarrow r=\sqrt{(64)^2+0}=64 \text{ and } \theta=\tan^{-1}\frac{0}{64}=0$$

At $\qquad k=0 \quad \Rightarrow \quad \text{1st root } w_0=\sqrt[6]{64}\, e^{i\left(\frac{0}{6}+0\times\frac{2\pi}{6}\right)}=e^0$

$=\sqrt[6]{64}\, e^0 = 2$

$\therefore \qquad z=2$

At $\qquad k=1 \quad \Rightarrow \quad \text{2nd root } \sqrt[6]{64}=2$

$w_1 = 2e^{i\left(\frac{0}{6}+\frac{2\pi}{6}\right)} = 2e^{i\frac{\pi}{3}}$

$\therefore \qquad z=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=1+i\sqrt{3}$

at $\qquad k=2 \Rightarrow \text{3rd root } w_2 = 2e^{i\left(0+2\times\frac{2\pi}{6}\right)} = 2e^{i\frac{4}{6}\pi} = 2e^{i\frac{2}{3}\pi}$

$\therefore \qquad z=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=-1+i\sqrt{3}$

at $\qquad k=3 \Rightarrow \text{4th root } w_3 = 2e^{i\left(0+3\times\frac{2\pi}{6}\right)} = 2e^{i\pi}$

$\therefore \qquad z=2(\cos\pi+i\sin\pi)=-2$

at $\quad k=4 \Rightarrow 5\text{th root } w_4 = 2e^{i\left(0+4\times\frac{2\pi}{6}\right)} = 2e^{i\frac{5}{3}\pi}$

$\therefore \quad z = 2\left(\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi\right) = -1 - i\sqrt{3}$

at $\quad k=5 \Rightarrow 6\text{th root } w_5 = 2e^{i\left(0+5\times\frac{2}{6}\pi\right)} = 2e^{i\frac{5}{3}\pi}$

$\therefore \quad z = 2\left(\cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi\right) = -1 - i\sqrt{3}$

$$\left(\textbf{Ans.} \pm 2; 1 \pm i\sqrt{3}; -i \pm i\sqrt{3}\right)$$

PROBLEM 9.12

Find the six Solutions of the equation: $z^6 + 2z^3 + 2 = 0$.

Solution:

$$z^6 + 2z^3 + 4 = 0$$

As, $\quad z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore \quad z^3 = \dfrac{-2 \pm \sqrt{4 - 4(1)(4)}}{2(1)} = \dfrac{-2 \pm \sqrt{4-16}}{2} = \dfrac{-2 \pm \sqrt{-12}}{2}$

$$= \frac{-2 \pm \sqrt{4}\sqrt{3}i}{2} = \frac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i$$

$$z = \sqrt[3]{-1 \pm i\sqrt{3}}$$

For $\quad \sqrt[3]{-1 \pm i\sqrt{3}} \Rightarrow r = \sqrt{(-1)^2 + 3} = \sqrt{4} = 2$

and $\quad \theta = \tan^{-1}\dfrac{\sqrt{3}}{-1} = \dfrac{2}{3}\pi$

$$k = 0 \quad \Rightarrow \quad w_0 = \sqrt[3]{2}e^{\frac{i}{3}\left(\frac{2}{3}\pi\right)} = \sqrt[3]{2}\, e^{i\frac{8}{9}\pi}$$

$$z = \sqrt[3]{2}\left(\cos\frac{8}{9}\pi + i\sin\frac{8}{9}\pi\right)$$

$$k=2 \Rightarrow w_2=\sqrt[3]{2}\,e^{i\left(\frac{2/3\pi}{3}+2\frac{2}{3}\pi\right)}=\sqrt[3]{2}\,e^{i\frac{14}{9}\pi}$$

$$z=\sqrt[3]{2}\left(\cos\frac{14}{9}\pi+i\sin\frac{14}{9}\pi\right)$$

For $\sqrt[3]{-1-i\sqrt{3}} \Rightarrow r=\sqrt{1+3}=2$ and

$$\theta=\tan^{-1}-\frac{\sqrt{3}}{1}=\frac{4}{3}\pi$$

$$k=0 \Rightarrow 4\text{th root}=w_3=\sqrt[3]{2}\,e^{i\left(\frac{4/3\pi}{3}\right)}=\sqrt[3]{2}\,e^{i\frac{4}{9}\pi}$$

$$z=\sqrt[3]{2}\left(\cos\frac{4}{9}\pi+i\sin\frac{4}{9}\pi\right)$$

$$k=1 \Rightarrow 5\text{th root}=w_4=\sqrt[3]{2}\,e^{\frac{i}{3}\left(\frac{4}{3}\pi+2\pi\right)}=\sqrt[3]{2}\,e^{i\frac{10}{9}\pi}$$

$$z=\sqrt[3]{2}\left(\cos\frac{10}{9}\pi+i\sin\frac{10}{9}\pi\right)$$

$$k=2 \Rightarrow 6\text{th root}=w_5=\sqrt[3]{2}\,e^{\frac{i}{3}\left(\frac{4}{3}\pi+4\pi\right)}=\sqrt[3]{2}\,e^{i\frac{16}{9}\pi}$$

$$z=\sqrt[3]{2}\left(\cos\frac{16}{9}\pi+i\sin\frac{16}{9}\pi\right)$$

$$\left(\textbf{Ans.}\sqrt[3]{2}\left(\cos\frac{2}{9}\pi\mp i\sin\frac{2}{9}\pi\right);\sqrt[3]{2}\left(-\cos\frac{\pi}{9}\mp i\sin\frac{\pi}{9}\right);\sqrt[3]{2}\left(\cos\frac{4}{9}\pi\mp i\sin\frac{4}{9}\pi\right)\right)$$

PROBLEM 9.13

Find all Solutions of the equation $x^4+4z^2+16=0$.

Solution:

$$x^4+4z^2+16=0$$

$$\Rightarrow \quad x^2=\frac{-4\mp\sqrt{16-64}}{2}=-2\mp2\sqrt{3i}$$

$$\Rightarrow \quad x=\sqrt{-2\mp2\sqrt{3i}}$$

For $\sqrt{-2\mp2\sqrt{3i}} \Rightarrow r=\sqrt{4+12}=\sqrt{16}=4$

and $$\theta = \tan^{-1}\frac{2\sqrt{3}}{-2} = \frac{2}{3}\pi$$

$$k = 0 \rightarrow \text{1st root} = w_0 = \sqrt{4}e^{\frac{i}{2}\left(\frac{2}{3}\pi\right)} = 2e^{i\frac{\pi}{3}}$$

$$\Rightarrow \qquad 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + i\sqrt{3}$$

$$k = 1 \rightarrow \text{2nd root} = w_1 = 2\,e^{\frac{i}{2}\left(\frac{2}{3}\pi + 2\pi\right)} = 2\,e^{i\frac{4}{3}\pi}$$

$$= 2\left(\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi\right) = -1 - i\sqrt{3}$$

For $\sqrt{-2 \mp 2\sqrt{3}i} \Rightarrow r = \sqrt{4+12} = 4$

and $$\theta = \tan^{-1}\frac{-2\sqrt{3}}{-2} = \frac{4}{3}\pi$$

$$k = 0 \rightarrow \text{3rd root} = w_2 = 2\,e^{\frac{i}{2}\left(\frac{4}{3}\pi\right)} = 2\,e^{i\frac{2}{3}\pi}$$

$$= 2\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right) = -1 + i\sqrt{3}$$

$$k = 1 \rightarrow \text{4th root} = w_3 = 2\,e^{\frac{i}{2}\left(\frac{4}{3}\pi + 2\pi\right)} = 2\,e^{i\frac{5}{3}\pi}$$

$$= 2\left(\cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi\right) = -1 - i\sqrt{3}$$

$$\left(\textbf{Ans. } \mathbf{1 \pm i\sqrt{3};\ -1 \pm i\sqrt{3}}\right)$$

PROBLEM 9.14

Solve the equation $x^4 + 1 = 0$.

Solution:

$$x^4 + 1 = 0$$

$$x^4 = -1 \Rightarrow x = \sqrt{-1} \Rightarrow r = \sqrt{1+0} = 1$$

and $$\theta = \tan^{-1}\frac{0}{-1} = \pi$$

$$\text{1st root} = w_0 = e^{\frac{i}{4}\pi}$$

$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$\text{2nd root} = w_1 = e^{\frac{i}{4}(3\pi)} = \cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$\text{3rd root} = w_2 = e^{\frac{i}{4}(5\pi)} = \cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$\text{4th root} = w_3 = e^{\frac{i}{4}(7\pi)} = \cos\frac{7}{4}\pi + i\sin\frac{7}{4}\pi = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$\left(\textbf{Ans. } \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \pm \frac{\boldsymbol{i}}{\sqrt{\mathbf{2}}}; -\frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \pm \frac{\boldsymbol{i}}{\sqrt{\mathbf{2}}}\right)$$