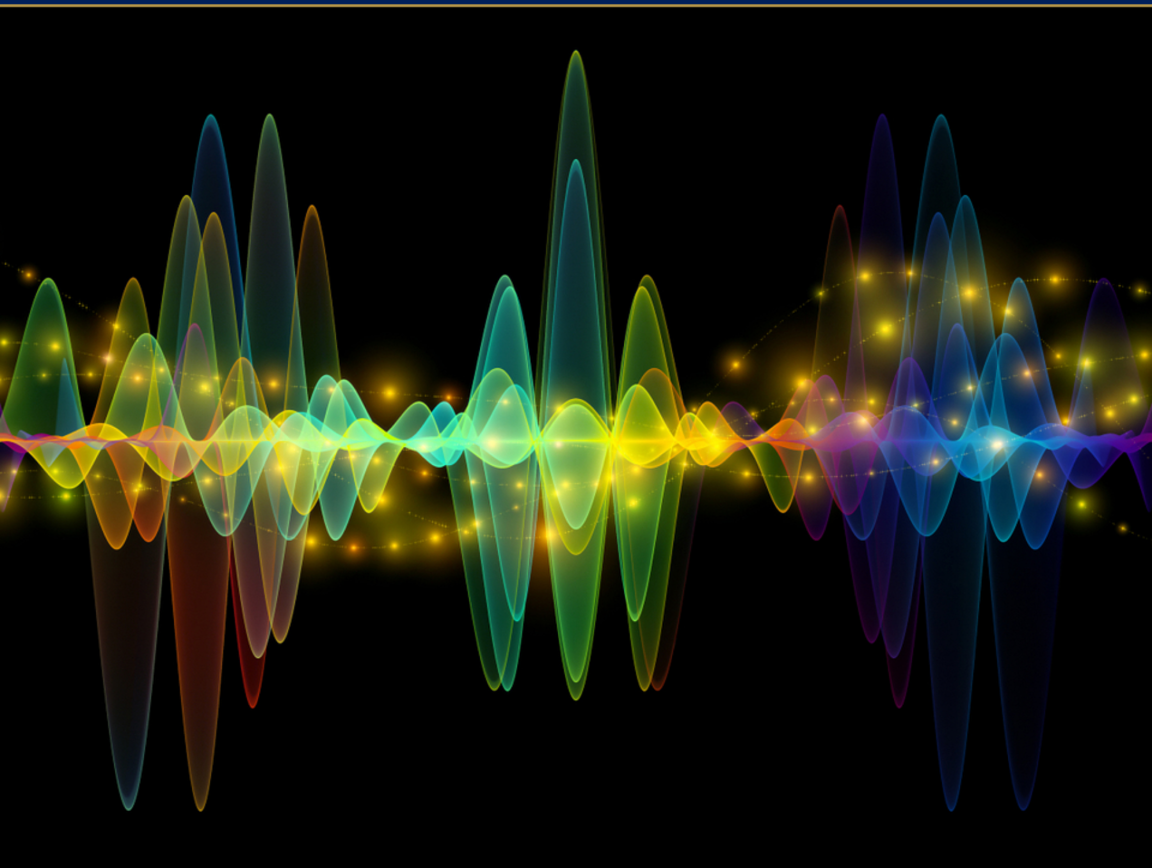
Newtonian Mechanics

A Modelling Approach

Second Edition



Derek Raine

Essentials of Physics Series

Newtonian Mechanics

Second Edition

NEWTONIAN MECHANICS

A Modelling Approach

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PREFACE

Newtonian mechanics is taught as part of every physics program for several reasons. It is a towering intellectual achievement; it has diverse applications; and it provides a context for teaching modelling and problem solving. I have tried to give equal prominence to all three missions in this text. To do this I have included some advanced material as well as the customary introductory topics. The book therefore is designed to be studied over an extended time-frame somewhat beyond the first year of a university physics program. This enables me to develop the problem-solving aspects more fully than in many other texts, as well as including some more advanced content. In particular I have tried to show how problems are approached in order to bring out the way one goes about constructing a solution or model. Tidy solutions and appropriate models rarely come fully formed, yet many texts present them as such, assuming that students will learn through their own trial and error. I think the trial-and-error process needs to be taught.

Each chapter begins with a problem, for which the following text provides the background to a solution. I hope that knowing what the question is makes the following material more digestible. I have included some end-of-chapter questions, but not drill exercises. These are so readily available (and constructible) that it seemed an extravagant use of paper to write yet more. The text itself contains some solved drill exercises which help to illustrate a particular concept.

The level of mathematics varies through some of the chapters. The more difficult sections can be omitted on first reading. I have assumed that students will be taking a parallel course in mathematical methods, but the early parts of chapters use plug-and-chug verification to avoid overburdening the student. On the other hand, if we avoid mathematical sophistication entirely, it is not possible to reach the required level of skill to build up the modelling expertise that Newtonian mechanics is supposed to teach.

It will be clear to readers that my approach to many subjects is not entirely conventional. For example, Newton's third law is treated first, weight is introduced before mass, energy is introduced before the equations of motion. This last I do for the particular reason of making contact with contemporary physics: the physics of elementary particles is encapsulated (roughly speaking) in an expression for the (quantum) energy of the Universe, and their dynamics follows from this. It also makes direct contact with Hamiltonian mechanics, an understanding of which makes quantum mechanics a little less impenetrable.

Much of the material for this book was developed in collaboration. I am particularly grateful to Dr. Edwin Thomas who not only originated some of the problems but read an initial version of the text and helped in proofreading for accuracy. It goes without saying that any remaining errors are mine alone. Sarah Symons and Naomi Banks also made helpful suggestions.

Derek Raine
Leicester
March 2021

CHAPTER 1

MECHANICAL MODELS

1.1 INTRODUCTION

We observe that the world changes. At first most of these changes appear random, but then we begin to observe the regularity of day and night, the periodicities of the seasons, the flow of water, and the transforming effect of fire. We wonder if we can perhaps control some of these changes. Gradually, we learn that to exploit nature, we must first understand changes. Progress in understanding change means describing it and isolating regularities, it means that understanding a surface complexity in terms of deep simplicity. We might link the start of this endeavor to Plato's challenge to the academicians of Athens to understand the complex movement of the stars and the planets in terms of motion on interlinked circles. We might highlight the development of kinematics in Oxford and Paris in the thirteenth century, isolating the features of motion under constant acceleration and describing it graphically. We could note the complexity of Ptolemy's epicycles brought to order by Kepler's discovery of the elliptical motion of the planet Mars, and Galileo's experimental finding that bodies fall with constant acceleration. Or, we could begin with the laws of motion as synthesized by Newton from the work of Huygens and Descartes into a code that can unravel the motion of all bodies – unless they are moving near the speed of light or inhabit the micro-world of the atom. Wherever we start, this is above all a story of progress in stripping away the inessentials for the given purpose; in short, a story of how to make models of the world in the language of mathematics.

This book is about that story: as a history, it is one of the greatest narratives of human endeavor; and as current science, it is one of the most significant underpinnings of modern technology.

Let us begin, amusingly and totally unfairly, with a speech to the British Association for the Advancement of Science given by Dionysius Lardner in 1838. Lardner said "Men might as well project a voyage to the Moon as attempt to employ steam navigation against the stormy North Atlantic Ocean." One hundred and fifty years separated the accomplishment of the two events but neither was as impossible as he had predicted. We do not have any record of why Lardner thought we could not travel to the Moon, but we do know why he thought that steamships could not cross the Atlantic. He believed that the resistance of a ship increases with its size; so more coal is required to feed the boilers of the larger ship that produce the power to overcome the resistance. But the size of the ship then has to be increased to carry the extra coal, which in turn increases the resistance requiring ever more coal. Eventually, Lardner believed that the maximum range would be reached using (presumably) an infinite amount of coal in an infinitely large ship.

A little mathematics, and some knowledge of ship design, enables us to see how the problem is, in fact, overcome. In order to proceed, it is easier to imagine ourselves in the frame of reference of the ship (or, stated more simply, just imagine ourselves *on* the ship). Then, the resistance force on the ship is proportional to the rate at which it destroys the momentum of the sea that tries to flow past it. This is proportional to the transverse cross-sectional area of the ship. Let us take some length scale L to characterize the size of the ship, the width or length, for example. Then, we imagine the ship to grow proportionately as we increase L (e.g., multiply all lengths by factor 2). The area will increase as $\propto L^2$ with the scale, L (so by a factor of 4 if we double L). However, the amount of coal carried increases as the volume, which increases more rapidly ($\propto L^3$). Thus, larger ships are, in fact, *more* suited to long distances than small ones. We can do even better if we make the ship long and thin (which is why ships *are* long and thin): we can increase the volume, while the transverse area remains almost constant.

What do we learn from this? First, that we need mathematics, or at least mathematical ideas, to clinch an argument, not mere words. Second, that to apply mathematics we need to simplify the situation to retain only what is relevant: here, it does not matter what the ship is made of, or even how it differs from a cuboid; the properties of the sea are unimportant, other than that it flows. And, we can adopt a convenient point of view (from the ship or from the land) in assessing the problem: the outcome cannot depend on which frame of reference we choose.

1.2 MODELS

We are going to look at a systematic way of thinking about models in physics. Let us introduce this through another example. Consider the orbit of the Earth around the Sun. There are two agents involved here: the Earth and the Sun; they are, if you like, the players on the stage. The Sun is going to be an external agent: that is to say, its properties are going to be fixed and unaffected by the presence of the Earth. Its only role will be to exert a gravitational pull on the Earth. Our second agent, the Earth, will treat it as a point mass with the properties that it has a position and a velocity. The two agents interact through the gravity of the Sun, which falls off with the inverse square of the distance between the Sun and the Earth. With this set-up, we look for possible orbits of the Earth around the Sun which repeat – that is to say which the Earth – in this model will track year after year. The outcome, as you probably know, is that the Earth must move in an elliptical orbit with the Sun at a special point called the focus of the ellipse, which particularly depends on how the system was formed (i.e., on the initial conditions).

Is this what really happens? No. The Sun is not at rest – it too moves under the influence of the gravity of the Earth, the Earth is not a point, it is not spherical, also it spins and the pair interacts through solar radiation and the solar wind as well as through gravity. And, that is before; we have taken account of the influence of the other planets of the solar system. Some of these differences do not affect the orbit, but some do. The point of the model is that it allows us to investigate the effect of the hypothesis that gravity follows an inverse square law. No other law would provide us with the gross features of the orbits. The model can then be extended *under the same hypothesis* to see if we can account for the detailed departures of the orbit from a perfect ellipse by adding in the previously omitted details to a more comprehensive model. Once we have used these models to establish our hypothesis about the nature of gravity, this will become part of our knowledge of physics that will be used in any other situations where we need to model gravitational interactions, for example in other planetary systems: our models should be consistent and we develop a body of knowledge of the laws of physics to ensure this.

To complete the story, you may know that things work out pretty well for the inverse square law, but not exactly once Einstein comes on the scene. Einstein's general theory of relativity enables us to say that the hypothesis of the inverse square law is not exactly true – no model based on it will agree exactly with all observations of the motion of the planets. In Einstein's theory of relativity, there are, in effect, forces of gravity on the Earth (and on the other planets) that modify the inverse square law, and which do enable us to account for planetary motion precisely.[1]

We use Einstein's theory to calculate departures from Newtonian gravity in any model of bodies orbiting under gravity. So let us think about the orbit of a GPS satellite around the Earth. To calculate this, we would need a model of the Earth. There is a number to choose from:

1. The Earth is a uniform sphere.
2. The Earth is a non-uniform sphere with density varying with radius.
3. The Earth is an ellipsoid.

[1] Even Einstein's theory may not be the final word: string theories, for example, suggest that there may be higher-order corrections to the equations of general relativity, although these would have a negligible effect on the solar system.

4. The Earth is a body the mass distribution (and shape) of which has been mapped (to some level of accuracy).
5. The Earth is exactly the shape and density of the Earth (the real world "experiment").

All (except the last) are approximations. Whether they are useful depends on what we want to do. Which model is the most appropriate to study the following? We will leave you to decide.

a) Satellite orbits
b) Earthquake determinations of the structure of the core
c) Tidal forces
d) Weather prediction
e) Solar system models.

1.3 ESTIMATES

Before we tackle a problem in detail, it is important to build an approximate model to get a rough idea of what to expect. Here is a historical example.

Newton attempted to test his postulate of the universal inverse square law of gravity by estimating the gravity required to keep the Moon in orbit. A body orbiting in a circle at radius R moves through a distance of order R in a quarter of a period ($T/4$), so the acceleration (distance per unit time) due to gravity at the body is *approximately* $R/(T/4)^2$. Comparing the acceleration due to gravity produced by the Earth at its surface ($g = 9.81$ m s^{-2}) with the gravity of the Earth at the Moon (g_m), we have therefore

$$\frac{g}{g_m} = 9.81\frac{T^2}{16R_m}.$$

The Moon is at a distance $R_m = 4 \times 10^8$ m and its orbital period is a month ($= 2.5 \times 10^6$ s), so we have $g/g_m \sim 10^4$.

What would the inverse square law give us? If $g \propto 1/R^2$ then $g/g_m = (R_e/R_m)^2$ where R_e (=6400 km) is the radius of the Earth, so we should have $g/g_m = (R_e/R_m)^2 \sim 5 \times 10^3$. This is a factor 2 out, not bad for our rough estimate.

Newton used a much better estimate for the acceleration of the Moon, but a rather worse estimate of the distance to the Moon, with the result that for several years, he did not believe the inverse square law to be exact. With a better knowledge of the distance to the Moon, the numbers worked out and Newton went on to write the *Principia.*

Two important points to remember in making estimates: quantities raised to high powers need to be known fairly accurately to get a good estimate; on the other hand only rough values are needed for quantities raised to fractional powers. Also, if a quantity is bounded by a large range, then the geometric mean is the best estimate for that quantity. For example, a useful estimate of a quantity that varies between 1 and 100 is usually not 50.5 (the arithmetic mean) but $\sqrt{1\times100} \sim 10$ (the geometric mean).

It is useful to practice using approximate models and approximate values to obtain the order of magnitude estimates. Here are some examples: which of the following are true?

a) 1 foot = 1 light nanosecond. (The speed of light is 3×10^8 m s^{-1}.)

b) ½ degree ~ the angle subtended by a penny coin at arm's length.

c) A piece of paper folded 25 times could stretch to the Moon.

1.4 UNITS AND DIMENSIONS

In the SI system, the standard base units in mechanics are the meter, kilogram, and second, corresponding to the dimensions of mass [M], length [L], and time [T]. Apart from the need to attach units to physical quantities, the dimensions of derived quantities are useful in several ways.

Dimensions have to balance in an equation, a fact which often allows one to check an equation – provided the equation is written with all the physical quantities in symbols and not subsumed in numerical values.

Dimensions also allow us to associate various physical meanings to a quantity. For example, force is mass times acceleration so has dimensions $[F] = MLT^{-2}$. This can be written as MLT^{-1}/T, that is, as the rate of change of momentum (because it is mass times velocity – or momentum – per unit time). Similarly, pressure is force per unit area, so has dimensions $[P] = [F]/L^2 = ML^{-1}T^{-2}$. This can be written as ML^2T^{-2}/L^3 or energy ($\propto mv^2$) per unit volume. This can be quite useful if one wants to estimate pressure.

For example, the pressure at the center of the Sun supports the Sun against its own gravity, so the energy per unit volume must be roughly equal to the gravitational energy. In Chapter 6, we shall see that the gravitational potential energy can be estimated as GM^2/R^4, where M is the mass of the Sun, R its radius, and G Newton's gravitational constant. Putting in values for the solar mass and radius, we find that the pressure at the center of the Sun must be of order 10^{14} N m^{-2}. This is a remarkable result: we have used a little mathematical physics to construct a "device" that "measures" the pressure at the center of the Sun. (Actually, we could go further: this pressure must also be roughly the energy density of the solar plasma, from which we could estimate the temperature of the solar interior.)

Finally, one can sometimes use dimensional analysis to extract the dependence of one physical quantity on others. For example, the drag of a body in a fluid must have the dimensions of a force and must depend on the area of the body, A, its speed v (a body at rest experiences no drag), and the density of the medium ρ (at low enough density the medium may as well not be there). The only combination of A, v, and ρ that has the dimension of a force (MLT^{-2}) is $A\rho v^2$. Of course, the shape of the body will add a numerical factor. In addition, there would be a viscous drag on the body, which can also be estimated by dimensional considerations, up to a numerical factor.

The disadvantage of units is that there are many different ones in use for the same quantity. This is partly historical and partly, sometimes, for the convenience of using numerical values as close as possible to order unity. This being so, it is often necessary to convert between units. There are various algorithmic ways of doing this. For example,

$$\begin{aligned} V\,(\text{miles h}^{-1}) &= V\,(\text{m s}^{-1}) \times (\text{miles m}^{-1}) \times (\text{s hr}^{-1}) \\ &= V\,(\text{m s}^{-1}) \times (8/5 \times 1000)^{-1} \times 3600 \text{ miles hr}^{-1} \end{aligned}$$

because there are 8/5 × 1000 m in a mile and 3600 s in an h. Note that meters (m) and seconds (s) cancel from the intermediate formula. Your speed in miles per second will be less than that in meters per second by a factor of the number of meters in a mile (divide by 8/5 × 1000) and your speed in meters per hour will be greater than that in meters per second by the number of seconds in an hour (multiply by 3600).

1.5 EQUATIONS

Estimates inform mathematics as well as numerical calculations. The most important aspect, once one has learned to work with symbols and not numerical values, is to learn to neglect small quantities. Let us look at some examples.

Suppose we put a girdle around the Earth, that is we wrap a rope around its circumference. Suppose the rope has a length $L = 2\pi R + \delta$ (where R is the radius of the Earth). How high off the ground is it? Most people would guess that the height is very much less than δ because the extra δ has to stretch all the way around the world.[2] This is a not very interesting question from a practical point of view, but let us see how the mathematics works out using a simple model of a spherical Earth. Let the height off of the ground be h. We have:

$$2\pi\left(R + h\right) = 2\pi R + \delta$$

[2] The philosopher Ludwig Wittgenstein liked to quote this as an example where a mental picture of the relation between big and small leads us astray: pouring a glass of water into the ocean does not have much effect on sea level. Quoted in, for example, *Wittgenstein and the Philosophy of Mind*, Ed Jonathan Ellis & Daniel Guevara (2012) OUP.

so

$$h=\frac{\delta}{2\pi}.$$

Thus, adding a meter to the length gives a height of about 15 cm – on whatever planet you choose!

Why has it worked out like this? Dividing through by $2\pi R$, we can write the equation another way:

$$1+\frac{h}{R}=1+\frac{\delta}{2\pi R}.$$

In other words, a 1% change in the circumference (δ/R) produces a roughly $\frac{1}{2\pi}\sim\frac{1}{6}$% (proportionate) increase in the radius (h/R) because the radius and circumference are linearly related. Put this way, the answer is entirely reasonable.

Consider next a completely different problem. What rise in sea level would result from a 1 degree rise in sea temperature? What model shall we choose? The simplest one, which we shall take as our starting point, is a sphere of radius R covered to a uniform depth h in a thin layer of water. Suppose that the coefficient of expansion of water is α. Then, the change in volume of the sea on expansion is α times the original sea volume:

$$\frac{4}{3}\pi(R+h+\delta)^3-\frac{4}{3}\pi(R+h)^3=a\left[\frac{4}{3}\pi(R+h)^3-\frac{4}{3}\pi R^3\right] \quad (5.1)$$

This looks like a lot of work; however, our model has both $h \ll R$ and $\delta \ll R$. So, expanding the brackets and canceling, we can approximate:

$$4\pi\,(R+h)^2\delta \sim 4\pi R^2 h\, a \quad (5.2)$$

neglecting the extra terms with powers of δ and h higher than the first, or

$$\delta \sim ah \quad (5.3)$$

since $h \ll R$.

Equation (5.1) makes it look as if the final answer should involve the radius of the Earth, R. The result (5.3) shows that the relative rise (δ/h) is independent of the radius of the planet. Is this reasonable? We cannot make a dimensional argument here, because there are too many lengths involved: the final answer could have been multiplied by any number of factors of h/R. The easiest way to see that the result is reasonable is to imagine a strip of water from around the circumference laid out (approximately) on a flat surface. Then, it does not matter how *long* the strip is: the rise in *height* will always be the same when it expands. Another way of seeing this is to compare it to putting a girdle around the Earth: the extra height (radius) is accommodated by a proportionate increase in length (circumference) without reference to the radius of the planet.

Note that we could have written down Equation (5.2) immediately by approximating the volume of a thin covering of the ocean on a sphere as area × depth. So, this is another check on the model.

For our final example, we look at the fall-off of pressure with height in the Earth's atmosphere. Suppose that a student, asked to estimate the height of the atmosphere, claims that the inverse square law of gravity means that gravity gets weaker as you get to greater heights in the atmosphere, and hence, that the top of the atmosphere is where gravity is so much weaker than it cannot stop the air molecules escaping. What do we make of this?

Of course, to be fair it all depends on what you mean by the top of the atmosphere, but we can agree that what most people mean by a significant atmosphere does not extend as far as low Earth orbit at a few hundred kilometers. (It is actually much less the FAI[3] defines the boundary between the atmosphere and outer space as the Karman line at 100 km.) We can see that the student's answer must be wrong with just a little appreciation of mathematics. The acceleration due to gravity, g, falls off with radius from the center of the Earth as an inverse square: $g \propto 1/R^2$. The only length scale in the gravitational model is the radius R. (The presence of the atmosphere does not alter this: gravity is essentially unaffected by the atmosphere.) So R is the length scale on which gravity gets significantly weaker, a

[3] Fédération Aéronautique Internationale.

scale very much greater than the height of the atmosphere. Thus, as far as the atmosphere is concerned we can treat g as approximately constant. The explanation for the thinness of the layer of atmosphere around the Earth must lie elsewhere.

Another way of looking at this is to work out how much g changes by over a height $h << R$. We do *not* do this by tapping numbers into a calculator. Instead, we derive a feeling for the way g falls off by expanding the inverse square law for $h << R$:

$$g \propto \frac{1}{(R+h)^2} \sim \frac{1}{R^2} - \frac{2h}{R^3},$$

using the binomial theorem $\left((x+\delta)^{-2} = x^{-2}\left(1 - \frac{2\delta}{x} + \ldots\right) \text{for } \frac{\delta}{x} \ll 1 \right)$ and neglecting terms in higher powers of h/R. So close to the surface, g falls off linearly with height.

The Earth is 6400 km in radius, so if the atmosphere were to extend this by as much as 200 km, it would amount to no more than 3%. Gravity is an inverse square law, so a 3% increase in radius means roughly a 6% decrease (double 3%) in gravity: scarcely noticeable. The atmosphere would extend by several Earth radii if the explanation given were really true. In fact, the height of the atmosphere is governed by the amount of air, and the way pressure falls off with height in an approximately constant gravitational field and has a true scale height (height to fall by a factor $1/e$) of around 8 km.

1.6 CHAPTER SUMMARY

- Physics in general, and mechanics in particular, involves making mathematical models of the world.
- A model seeks to simplify reality as much as possible for the purpose to which it is being put.
- A model is defined in terms of the agents and their interactions. Simplification, therefore, means identifying the significant agents and their essential interactions.

- Models in mechanics should be described in terms of mathematical symbols for dimensional quantities with the entry of numerical values reserved for the final step. This allows dimensions to be checked for consistency.
- The mathematics should be approximated appropriately to the model, especially in the neglect of small quantities where justified. This enables the results to be interpreted more readily.
- The result of a model should be expressed and explained in words (and/or graphically) and examined to check that it is reasonable.

CHAPTER 2

FORCES

In this chapter, we are going to address the following problem:

Problem 1: The figure shows a horse and cart. In due course, the farmers will have had enough of being photographed and will want to transport their harvest to market or storage. How does the horse pull the cart?

What are the forces in and on the system of horse and cart; why do these forces move the cart in some circumstances but not others (e.g., if the cart is too laden)?

Picture credit: *http://www.flickr.com/photos/hartlepool_museum/5933914248/sizes/z/in/photostream/*

2.1 ACTION AND REACTION

We start by considering various cases where the forces on a body are in equilibrium, hence where the forces do not change the state of motion of the body.

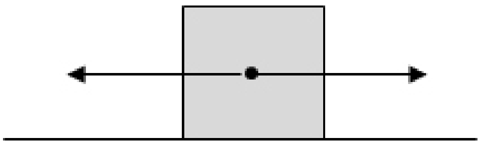

Figure 2.1: Horizontal forces on a block at rest on a horizontal plane

Consider a block at rest on a flat plane as in Figure 2.1. We imagine that the block is subject to equal and opposite horizontal forces acting through a common point, as indicated by the arrows. By symmetry, the block cannot move. If the forces on a body do not change its state of motion, we say that the forces are in *equilibrium*. This suggests that a body that does not move must be acted on by equal and opposite forces in both magnitude and direction, hence must be subjected to no net force (or no forces at all).

If forces of the same magnitude in Figure 2.1 were not to act through a common point, we should have a more complicated situation in which the block could tip over. We shall deal with this later: for the moment, all forces on an extended body are assumed to act through a common point. Alternatively, we can consider the body to be a point particle with no extension, so that all forces on it act through the same point by construction.

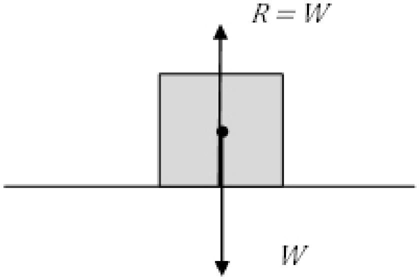

Figure 2.2: Vertical forces on a block at rest on a horizontal plane. The reaction force R is equal and opposite to the weight W

Consider now a block at rest on a flat plane as in Figure 2.2. If we were to imagine ourselves in the role of the plane, for example, by holding the block in our hand, we would feel the block pushing down. We attribute this to the *weight* of the block. Let us call this force W.

Thus, the block has a weight W, which is the force acting down on the plane. Experience shows that an unbalanced force causes an object to move. So we expect that the plane must act back on the block with a force equal and opposite to the weight of the block. In fact, if we imagine ourselves now in the role of the block, we feel this reaction as our weight. This is shown in Figure 2.2, where each force is represented by an arrow that points in the direction of the force and has a length proportional to the magnitude of the force.

The SI unit of force is the Newton (N), where $1\ \text{N} = 1\ \text{kg m s}^{-2}$.

Actually, in general, everyday experience alone does not always show that an unbalanced force causes an object to move. In one of the earliest systematic considerations of the issue, Aristotle pointed out that a man cannot move a ship.[1] It was Newton's insight to argue that the reason for this was not, directly, the weight of the ship, but the resistance offered by the water. Thus, even in this case, the ship does not move perceptibly because the forces on it are balanced. More than that, Newton proposed that in all cases, an action is balanced by an equal and opposite reaction – even when the reaction is not obviously visible. Thus, we have

Newton's Third law:

To every Action there is an equal and opposite Reaction.

Note that the law refers to the action and reaction between two agents (the block and the plane above): the action of agent A on B is equal and opposite to that of B on A. Each agent is acted on by the respective reaction.

There is a lot of confusion on the issue of action–reaction pairs and you may well have been told that what you have just read is wrong. The reason offered is that action–reaction pairs have to be of

[1] Since water has no static friction, in principle, a man should be able to move a ship ever so slowly, but not, at least in Aristotle's experience, perceptibly.

the same type. So, for example, the gravitational force of the Earth on the block must be equal to the gravitational force of the block on the Earth and that the electrostatic force of the block on the plane is balanced by the electrostatic force of the plane on the block. These statements are true: each of these pairs of forces have to be equal in all circumstances. But when we observe that a block is in fact at rest on a plane, we are under no compunction to consider the origin of weight or of the material forces. We are free to restrict the system of interest to us to the block and the plane. In that case, it is a matter of contingent truth (*if* the system is in equilibrium) that the action of the weight is balanced by the reaction of the plane.

2.2 FORCES IN EQUILIBRIUM

We now put this together in Figure 2.3. The block is again at rest by symmetry. The forces form a closed figure.

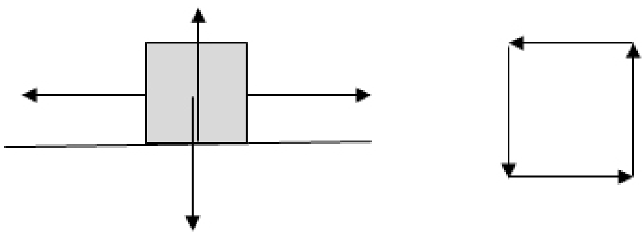

Figure 2.3: Forces on a block at rest on a horizontal plane. In equilibrium, the forces acting through a point form a closed figure

Experimentally, we find that if the forces acting through a point are applied at an angle (rather than horizontally and vertically), they will still balance if they form a closed figure.

Figure 2.4 shows another example of three forces applied at an angle to a body. The forces balance since they form a closed figure. The diagram shows why this is. The forces are labeled B, R, and G. The arrows are drawn in the direction of the forces and their lengths represent the magnitude of the forces. Horizontally, the force G to the left is balanced by the force R to the right and vertically, the forces R and G are balanced by B. This balance will always be the result if the forces form a closed figure.

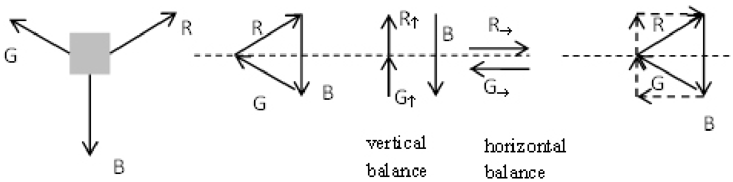

Figure 2.4: Forces in equilibrium form a closed figure. The horizontal (→) and vertical components (↑) are indicated by subscripts. The components of forces in horizontal and vertical directions balance and hence balance in any pair of orthogonal direction. This implies that forces behave as vectors

A necessary condition for a body to be in static equilibrium is that the forces on it balance. Forces balance if they form a closed figure.

This is a necessary condition only (it must be satisfied by bodies in equilibrium, but it is not sufficient to guarantee that a body is in equilibrium) because we have assumed that the forces act through a point.

The fact that forces forming a closed figure balance is equivalent to the *parallelogram law of force* and hence to the fact that force is a *vector*, that is, it behaves "like an arrow on the page" having a length and direction. To see this, consider the triangle of forces in Figure 2.5. Since the triangle is a closed figure, the forces balance, so the net force must be zero. Thus (Figure 2.5a)

$$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0.$$

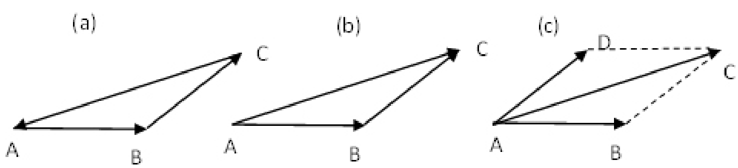

Figure 2.5: The fact that forces in equilibrium form a closed figure is equivalent to the parallelogram law

Equivalently (Figure 2.5b)

$$\overrightarrow{AB}+\overrightarrow{BC}=-\overrightarrow{CA}=\overrightarrow{AC} \tag{2.1}$$

Equation (2.1) is the parallelogram law in Figure 2.5c, since $\overrightarrow{BC}=\overrightarrow{AD}$:

$$\overrightarrow{AB}+\overrightarrow{AD}=\overrightarrow{AC}.$$

The vector $\overrightarrow{AC}$ is called the *resultant* of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

Figure 2.6 also shows how a force can be broken into *orthogonal components*. To express this algebraically, as well as in pictures, we introduce vectors **i** and **j**, each of unit length, in the horizontal and vertical directions, respectively (Figure 2.6).

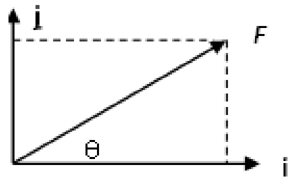

Figure 2.6: The decomposition of a force into components. The vectors **i** and **j** are orthogonal unit vectors (i.e., are perpendicular to each other and have unit length)

From Figure 2.6, we can then write for a force $\boldsymbol{F}$, with magnitude F:

$$\begin{aligned} F &= F\cos\theta\,\mathbf{i} + F\sin\theta\,\mathbf{j} \\ &= F_x\mathbf{i} + F_y\mathbf{j} \end{aligned} \tag{2.2}$$

In three dimensions, we again find that forces in equilibrium form a closed figure. To obtain the analog of Equation (2.2), we introduce a third unit vector **k** orthogonal to **i** and **j**, whence (Figure 2.7):

$$\begin{aligned} \boldsymbol{F} &= F\sin\theta\cos\phi\,\mathbf{i} + F\sin\theta\sin\phi\,\mathbf{j} + F\cos\theta\,\mathbf{k} \\ &= F_x\mathbf{i} + \mathrm{F}_y\mathbf{j} + F_z\mathbf{k} \end{aligned}$$

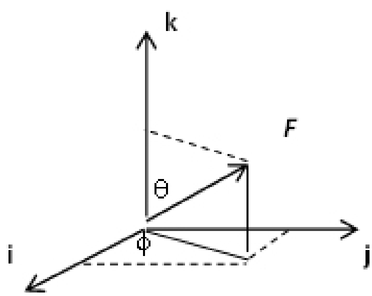

Figure 2.7: A vector *F* in three dimensions in terms of its components

We can now see the relation between the condition that balanced forces form a closed figure, and that the components of the net force in any direction are zero. If the forces form a closed figure, then the vector sum is zero; hence,

$$\sum F = \sum F_x \mathbf{i} + \sum F_y \mathbf{j} + \sum F_z \mathbf{k} = 0,$$

from which

$$\sum F_x = \sum F_y = \sum F_z = 0,$$

and hence the components in the directions of the axes sum to zero. But the axes can be chosen in any three orthogonal directions; hence, for a body in equilibrium, the components of force in any directions sum to zero.

Example: What is the resultant of a force $F_1 = 5$ N and a force of $F_2 = 2$ N acting in directions an angle 30° apart (Figure 2.8)?

Solution (1): The parallelogram law

The cosine rule gives

$$F^2 = F_1^2 + F_2^2 - 2F_1F_2 \cos 150^\circ$$
$$= 25 + 4 + 20 \times \frac{\sqrt{3}}{2} = 46.3$$

So F = 6.8 N. The sine rule then gives

$$\sin\alpha = 2 \times \frac{\sin 150^\circ}{6.8} = 0.15$$

and α = 8.5° to F_1.

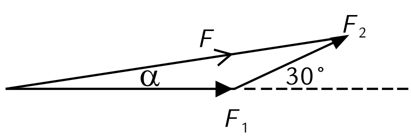

Figure 2.8: Addition of forces

Solution (2): Resolving forces

Since we can choose the axes arbitrarily, let $\boldsymbol{F}_1 = 5\mathbf{i}$; then

$$\boldsymbol{F}_2 = 2 \cos 30^\circ\, \mathbf{i} + 2 \sin 30^\circ\, \mathbf{j},$$

and

$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 = \left(5 + \sqrt{3}\right)\mathbf{i} + \mathbf{j}.$$

Thus, $F^2 = \left(5+\sqrt{3}\right)^2 + 1 = 25 + 10\sqrt{3} + 3 + 1 = 37.66.$

Then $\cos\alpha = \dfrac{F.F_1}{|F_1||F|} = \dfrac{25+5\sqrt{3}}{6.8\times 5} = 0.99$ so $\alpha = 8.1^{\circ}$.

The two equivalent methods appear to yield slightly different results! Which is the more accurate? Near $\alpha = 0$ the cosine is changing slowly so we need a very accurate value for $\cos\alpha$ to get an accurate value for α, whereas the inverse sin function is changing rapidly, so gives greater accuracy. Keeping three decimal places in solution 2 also gives $\alpha = 8.5^{\circ}$ in agreement with Solution 1.This is a useful lesson (but definitely does not mean that you should indulge in the rookie error of quoting all 32 decimal places on your calculator display).

2.3 HORSE BEFORE CART

Let us return to the horse and cart and trace all the action–reaction pairs as it starts to move off. First of all, assume that the brakes have been applied to the wheels so they are not free to turn. We will release them shortly once the horse is ready. Imagine yourself as the horse.

> Your weight is supported by the reaction on your hoofs from the ground. You push backwards on the ground and the ground responds by pushing you forwards with an equal force, assuming that friction is sufficient to prevent you from slipping. You pull on the cart and the cart pulls you back with an equal and opposite force.

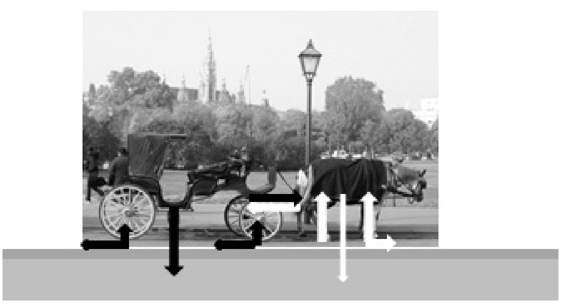

Figure 2.9: Forces on the horse (white arrows) and on the cart (black arrows). The forces on the Earth balance the vertical arrows (Picture: Microsoft Clip Art)

Now imagine yourself as the cart.

> Your weight is supported by the reaction on your wheels from the ground. The horse is pulling you forward and you are pulling back on the horse. The pull is communicated to the axle. The wheels cannot rotate because the brakes are on and they also cannot slip as long as the friction from the ground is sufficient to pin the points of contact to the ground. The horse and cart remain at rest.

To complete the picture, imagine yourself as the Earth.

> You are pulling on the horse and cart by your gravity and they are pulling on you equally by their gravity. Furthermore, they are pushing on you by their weight, compressing you slightly and you are pushing back with the pressure generated by the compression.

Note that the gravitational attraction of the Earth on the horse and cart is equal and opposite to the gravitational attraction of the horse and cart for the Earth, and that the compressional force on the horse and cart is equal and opposite to the vertical compressional force on the Earth. The effect of these forces on the Earth is small and we neglect them; the corresponding reaction forces on the horse and cart are respectively equal in magnitude to the corresponding actions. However, the effect of these forces is rather larger on the horse and cart because of the rather different mass of the horse and cart compared to the mass of the Earth. We will see why the magnitude of the effects differ when we come to Newton's second law.

Notice carefully that the forces on any stationary agent balance; these forces on a single agent do not constitute action and reaction pairs, which must act on different agents! Confusing this issue leads to the puzzle as to how the horse can pull the cart: if their action and reaction balance, so the argument goes, there is no net force to accelerate the cart from rest. In Figure 2.9, the white arrows indicate forces acting *on* the horse (the reaction of the cart and the ground) and the black arrows indicate forces acting on the cart (the action of the horse and the ground). Each will accelerate if the forces acting *on it* do not balance.

2.4 STATIC FRICTION

Before we release the brakes on the wheels, the cart will move off only if the force on it is sufficient to overcome the friction between the wheels and the ground and cause it to skid. How large is the frictional force? Experiment shows that the frictional force will be just large enough to balance the applied force on the cart up to a maximum given by

$$\boldsymbol{F}_{max} = \mu \boldsymbol{R}, \tag{2.3}$$

where $\boldsymbol{R}$ is the normal reaction of the ground on the cart. In the simple case that we have only the weight of the cart W pulling down, we have seen that $R = W$ (Figure 2.2). The quantity μ is called the *coefficient of friction*. Its value is usually in the range $0 < \mu < 1$, although values greater than unity are possible. Once the force on the cart exceeds this maximum, friction will no longer maintain the balance of forces. Note that the coefficient of friction μ depends on the materials in contact, but not on their apparent surface areas in contact (Section 2.8). This result was first published by Amontons in 1699.

2.5 SLIDING FRICTION

Clearly, it is only possible to move the cart with the brakes on if the friction when the wheels are sliding is less than when they are at rest, since otherwise sliding would produce a greater restraining force than the one we are supposing has been overcome. We therefore have to distinguish between sliding friction and static friction. Static friction is variable up to a maximum value. What about sliding friction? Suppose it were to have a constant magnitude. Then once the cart was in motion friction would never balance the applied force and the cart would continue to accelerate forever. Even if our experience with horse drawn vehicles is limited, our experience with motorized transport shows that this does not happen. Therefore, as the speed of sliding increases, so too must the friction: sliding friction is a *velocity-dependent* force. The functional dependence of the force on speed is something that has to be determined experimentally: all we know is that it must be an increasing function of speed. Simple models are $F_{sliding} \propto -v$ or $F_{sliding} \propto -v|v|$. (You might be tempted to write v^2 instead of $v|v|$, but if you do this you need to take care that you adjust the sign so that the friction always opposes the motion i.e. that the frictional force changes sign when the motion reverses.).

2.6 A FRICTION PARADOX

Before we leave friction, we need to address a question that puzzles a lot of students. Imagine a block moving at constant speed under the balance of an external force and an equal opposing frictional force. We know that friction causes a loss of energy as heat; so energy must be supplied to the block by the external force to keep its speed constant. This is common experience: we will return to the conceptual details later. Here is the puzzle: change your frame of reference by running with constant speed alongside the block. Now what you see is a block at rest with two equal and opposite forces in balance. So there should be no heat dissipation! How can we have removed the need to supply energy to the block just by viewing it from a different platform? Of course, this is not possible, so where is the fallacy?

To solve the puzzle, we need to understand the origin of friction. Friction cannot arise between exactly smooth surfaces: it requires fluctuations in the surfaces. In the final instance, these are provided by the atomic nature of matter, although in practice few surfaces are smooth down to the atomic scale. The roughness means that the interaction between the surfaces is not constant: friction cannot be a constant force. Only when we take a macroscopic average of the fluctuating force, do we get an apparently smooth behavior. If the frictional force is not exactly constant, then it cannot balance a constant external force moment by moment. Sometimes the body accelerates a little, sometimes it decelerates, maintaining a constant speed only on average and dissipating energy in the process. This behavior is the same whether we view it from a fixed point or from a uniformly moving platform. The fluctuating nature of frictional forces resolves the paradox.

2.7 ROLLING FRICTION

Let us return to the horse and cart and release the brakes. The wheels are now free to rotate. The frictional force now acts to stop the wheels from sliding. This means that the point of contact with the ground is prevented from sliding forward, so the friction still acts backward. As the wheels turn, they experience a rolling friction from the changing contact with the ground. This must be less than the sliding friction or the wheels would slide rather than roll. There is also a velocity-dependent sliding friction from the axle bearings, which now also contributes to prevent the horse from accelerating the cart to ever greater speeds.

2.8 CONTACT AREA

There is one more problem we need to clear up: how does friction depend on the area of contact between two surfaces? The paradox is this: according to the formula $F = \mu R$, the maximum frictional force between two objects depends only on the reaction on one object on the other and not on the apparent area of contact. So why would we increase the area of the tires on a car to increase the road holding?

The well-known example of the impossibility of separating two books with the pages interleaved appears to demonstrate precisely this dependence of the force on contact area. In fact, it does no such thing. The resolution of this paradox is that each page has the increasing weight of all the pages above it providing the frictional response. Thus, two 500 page books require a force 250 times greater than that between two sheets of paper, not because of the increased area but because of the weight of the pages.

Car tires are another story entirely: a skidding tire leaves a strip of rubber on the road, so this is not solid friction between two surfaces: it is harder to strip off a layer of rubber from a wider tire than from a narrower one.

We can understand this better if we consider that between two rough surfaces (hence, in practice, between any surfaces), the area of actual contact in a unit of visible area depends on the pressure. An increasing pressure deforms the surfaces leading to a greater area of contact, proportional to the pressure. The frictional force depends on the total number of points of contact, hence to the pressure times the area, or reaction force, between the surfaces, and not on the visible areas of overlap.

2.9 TORQUE: THE MOMENT OR COUPLE OF A FORCE

Our next problem is how do we explain that a wheel with equal and opposite forces (from the ground and the axle) nevertheless moves? The resolution will be that the forces in this case do not act through the same point.

Consider the simple situation shown in Figure 2.10. Two blocks of equal weight *W* are placed equidistant from the pivot point (or *fulcrum*) of a *lever*. We assume that the weight of the arm of the balance can be neglected.

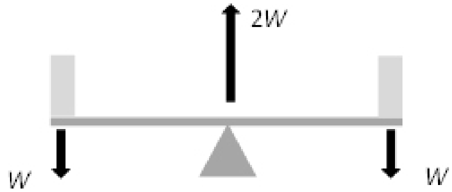

Figure 2.10: Two equal blocks at equal distances from the fulcrum of a lever. The upward reaction from the fulcrum balances the weights

Since the arrangement is symmetrical, there can be no preference for one of the blocks to move down and the other up, so the system does not move. Therefore, the forces on the lever arm must balance. If we now move one of the blocks toward the pivot, the forces will still balance but the arm will nevertheless tip. Observation shows that it tips down on the side with the weight further from the pivot.

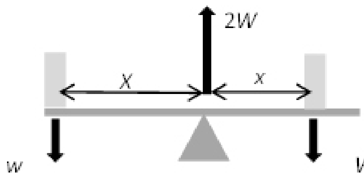

Figure 2.11: The arm will balance when *wX* = *Wx*. Under this condition, there is zero couple on the system

More detailed experiments show that for the weights to balance, we must have

$$wX = Wx. \tag{2.4}$$

Actually, of course, experiment can never tell us this exactly, but we assume that it to be exactly true and see where that leads.

First, we need to give names to the quantities in Equation (2.4). We select a point in space: any one will do, but usually a convenient one suggests itself. In this case, we choose the fulcrum. We then define the moment $\boldsymbol{G}$ of a force (or its couple or torque) about the chosen point to be the force multiplied by the perpendicular distance to the line of action of the force (Figure 2.12). In vectors, $\boldsymbol{G}$ is the vector product (or cross product)

$$\mathrm{G} = \mathrm{r} \wedge \mathrm{F}. \tag{2.5}$$

So $\boldsymbol{G}$ has a direction perpendicular to the plane containing the force and our chosen point about which it is acting and a magnitude rF.

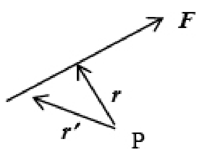

Figure 2.12: The moment ***G*** of a force about a point *P* is given by the vector product of the force and a vector from *P* to the line of action of the force

In fact, we can take the cross product of $\boldsymbol{F}$ with any vector from P to the line of action of $\boldsymbol{F}$, $\boldsymbol{r}'$ say, because

$$\boldsymbol{r}' \wedge \boldsymbol{F} = \left[\boldsymbol{r} + \left(\boldsymbol{r}' - \boldsymbol{r}\right)\right] \wedge \boldsymbol{F} = \boldsymbol{r} \wedge \boldsymbol{F}.$$

The final equality arises because $\boldsymbol{r}' - \boldsymbol{r}$ is parallel to $\boldsymbol{F}$ so $(\boldsymbol{r}' - \boldsymbol{r}) \wedge \boldsymbol{F} = 0$. If $\boldsymbol{r}$ is chosen to be perpendicular to $\boldsymbol{F}$, then the magnitude of the couple $\boldsymbol{G}$ is just $G = |\boldsymbol{G}| = rF$ in a direction normal to the plane of $\boldsymbol{r}$ and $\boldsymbol{F}$.

We now assert:

For a body to be in static equilibrium, the moments of the forces on it must balance (i.e., the net moment must vanish).

To show this assertion is meaningful, we have to show that if the moments balance about one point, they balance about any other. To do this, assume that forces $\boldsymbol{F}_i$ act at points $\boldsymbol{x}_i$ and that these forces are in equilibrium. Then, we have

$$\sum \boldsymbol{F}_i = 0,$$

and

$$\sum \boldsymbol{x}_i \wedge \boldsymbol{F}_i = 0.$$

Now change the origin to X. We have to show that

$$\sum (\boldsymbol{X} + \boldsymbol{x}_i) \wedge \boldsymbol{F}_i = 0.$$

But

$$\sum (\boldsymbol{X} + \boldsymbol{x}_i) \wedge \boldsymbol{F}_i = \boldsymbol{X} \wedge \sum \boldsymbol{F}_i + \sum \boldsymbol{x}_i \wedge \boldsymbol{F}_i = 0,$$

since each sum (of forces and moments) vanishes, which is what we set out to prove.

2.10 CONDITION FOR STATIC EQUILIBRIUM

We can now summarize our results. For a body in static equilibrium:

i) *the vector sum of the forces acting on the body must equal zero*

ii) *the moment of the forces on the body about any point must equal zero.*

Although we shall not show it, these conditions are also sufficient for a body to be in equilibrium, that is, if the forces on a body satisfy these conditions, then the forces must be in equilibrium. Thus, to solve a problem in statics, we set the sums of each component of the forces to zero and the moment to zero:

$$\sum F_x = \sum F_y = \sum F_z = 0,$$

and

$$\sum (\boldsymbol{r} \wedge \boldsymbol{F})_x = \sum (\boldsymbol{r} \wedge \boldsymbol{F})_y = \sum (\boldsymbol{r} \wedge \boldsymbol{F})_z = 0,$$

where the sums are over all the forces acting on the body. We call this *resolving and taking moments*.

2.11 CENTER OF GRAVITY

It is convenient to remove the lever arm or balance now and consider the abstract point about which the moments of a system of weights would be zero. We call this the *center of gravity*. For two weights w and W, it is defined by dividing the line between them in the ratio $X{:}x$ such that

$$wX + Wx = 0. \tag{2.6}$$

In general, it is the point about which the moment of the weights is $\boldsymbol{G} = 0$.

We can now describe why the lever balances under the *Archimedean condition* (2.4): this condition ensures that the center of gravity coincides with the fulcrum. Thus, the overall forces on the beam and the weights (gravity and the reaction from the fulcrum) not only balance in magnitude and direction but also in point of application.

Example: Center of Gravity

The most comfortable way to carry an object across your shoulder is to balance it at the center of gravity. Find the center of gravity of the spade in Figure 2.13. The handle is a cylinder of length h = 0.8 m weighing W = 8 N, and the blade is a rectangle of length b = 20 cm weighing w = 5 N.

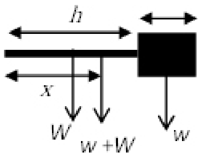

Figure 2.13: A model of a spade

Solution: If the center of gravity is a distance x from the top of the handle, then taking moments of the top of the handle

$$(W+w)x = w\left(h+\frac{b}{2}\right)+W\frac{h}{2},$$

from which

$$\begin{aligned} x &= \frac{5}{5+8}\left(0.8+\frac{0.2}{2}\right)+\frac{8}{5+8}\left(\frac{0.8}{2}\right) \\ &= 59\,\text{cm}. \end{aligned}$$

2.12 AN EXAMPLE

A traditional example of a statics problem that is tackled by resolving forces and taking moments is that of ladder against a wall. The figure shows the forces involved, where W, the weight of the ladder, acts through the center of gravity of the ladder, which we assume to be the midpoint. The ladder has length $2l$.

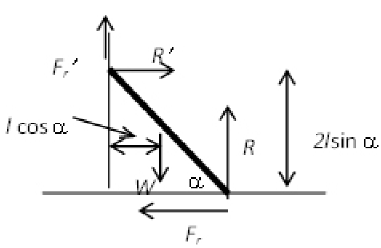

Figure 2.14: A ladder against a wall

We solve the problem by resolving the forces and taking moments.

Resolving horizontally:

$$F_r = R'; \tag{2.7}$$

and resolving vertically:

$$F_r' + R = W. \tag{2.8}$$

Taking moments of the point of contact with the wall:

$$Wl\cos\alpha + 2F_r l\sin\alpha = 2Rl\cos\alpha. \tag{2.9}$$

Consider first the case in which the wall is frictionless (i.e., when the friction from the wall is much less than that from the ground). We get

$$F_r = \mu R = \mu W \tag{2.10}$$

at the point of slipping. So from (2.9),

$$W\cos\alpha + 2\mu W\sin\alpha = 2W\cos\alpha,$$

or

$$\tan\alpha = \frac{1}{2\mu}. \tag{2.11}$$

For stability then, we require $\alpha > \tan^{-1}(1/2\mu)$. Once again we try out a numerical example to check this is reasonable. For example, if $\mu = 0$, so all the surfaces are frictionless, we require $\alpha > \pi/2$, which is impossible, as it should be the ladder cannot stay against the wall without friction. At the other extreme, if $\mu = 1$, then $\alpha = \tan^{-1}0.5$, so the ladder cannot rest at a shallower angle than about 27°.

Let us now introduce friction on the wall, so at the point of sliding $F_r' = \mu' R'$. Then, using (2.8),

$$F_r = \mu R = \mu\left(W - F_r'\right) = \mu\left(W - \mu' R'\right) = \mu W - \mu\mu' F_r,$$

so

$$F_r = \frac{\mu W}{1 + \mu\mu'}. \tag{2.12}$$

Then from Equations (12.3) and (2.7), we have

$$2F_r \sin\alpha = (W - F_r')\cos\alpha = (W - \mu' R')\cos\alpha = (W - \mu' F_r)\cos\alpha.$$

A little algebra using (2.12) gives

$$\tan\alpha = \frac{1 - \mu\mu'}{2\mu}.$$

Thus, the friction on the wall reduces the angle at which the ladder slips, as you would expect. If both $\mu = 1$ and $\mu' = 1$, then the ladder will not slip at all. Note how the mathematics gives a precise answer if we need it, but we also check against special cases where we think we know the result to see if the model and the calculation are correct.

2.13 PROBLEM SUMMARY

To summarize, we are interested in the motion of the horse and cart together, so this is our agent. We are not interested in the reaction back on the Earth, so this provides the environment; that is, to say, it can act on the agents, but we are not concerned with their action back on it. The horse and cart are at rest while the forces and couples on them balance, or, equivalently, while the net force and net couple are zero. Both forces and couples are vectors so their components, taken in convenient directions, must balance. Thus, the system will remain at rest while the net vertical and horizontal forces are zero. Vertically, Newton's third law tells us that if there is no vertical acceleration, then the reaction from the ground equals the weight of the combined system. The reaction will be distributed between the wheels of the cart and the horse's hooves. To determine at which point a hoof slips or a cart wheel slides, we should need to know how the reaction is distributed, because for each point of contact, with reaction force R the horizontal friction cannot exceed μR.

Horizontally, the reaction of the ground on the horse's hooves provides the force to propel the system forward. Rolling friction with the ground and sliding friction in the bearings resist this motion.

Once the horizontal forces balance, the motion is again at a steady speed.

We can divide the system into parts if we wish – and this might be important if we are considering the strength of the coupling between the horse and the cart for example – but as far as the system as a whole is concerned Newton's third law guarantees that the internal forces cancel. Consider the schematic diagram in Figure 2.15 representing the horizontal forces on the horse and cart.

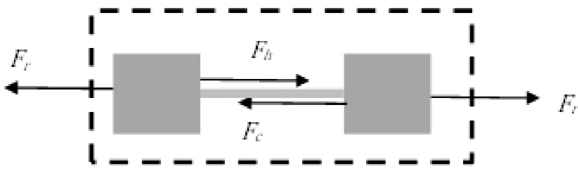

Figure 2.15: A schematic diagram of the forces on the horse *H* and cart *C*. F_r is the frictional force from the ground; F_h is the force the horse exerts on the cart; F_c is the force exerted by the cart on the horse and F_r' is the friction on the cart.

Looking at each agent, the cart and the horse, separately, we have for the static equilibrium of the horse

$$F_r = F_c,$$

and for the cart,

$$F_r' = F_h.$$

Newton's third law gives us

$$F_h = F_c,$$

from which we can deduce that

$$F_r = F_r'.$$

This is the condition for the static equilibrium of the system as a whole, indicated by the dashed rectangle in Figure 2.15. The cancellation of internal action–reaction pairs, Newton's third law, guarantees that we can divide the system into parts in any convenient way.

Later, we shall see how Newton's other laws enable us to calculate the acceleration of a system in which the forces do not balance.

Problem 2: For the second part of this chapter, we are going to address the following problem:

The figure shows the Great Pyramid of Giza together with a pictorial illustration of one idea for the way in which it was built, known as the ramp theory. According to this view, the building stones were hauled up the ramps. The theory is disputed by scholars because the ramps must be shallow and this implies that a vast amount of material is required to build the ramps. Some remains of ramp-like structures are known, but the absence of more visible evidence makes the theory problematic.

Clearly, a crucial aspect of the theory is the angle of the ramp: the stone blocks typically weigh 60 tons wt; up what angle is a team of men likely to be able to haul a stone block?

Great Pyramid of Giza, also known as the Pyramid of Khufu or Pyramid of Cheops (*http://www.lifeslittlemysteries.com/2174-cost-build-great-pyramid.html* CREDIT: Nina | Creative Commons *http://www.cheops-pyramide.ch/khufu-pyramid/ramp-models.html*)

2.14 INCLINED PLANES

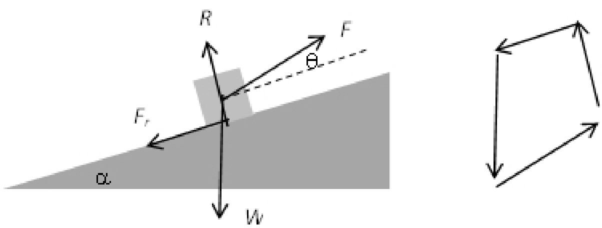

Figure 2.16: The forces acting on a block on an inclined plane

Our problem is to determine the optimum angle at which to pull a block up an inclined plane by a rope. We shall approach the problem in stages, including some false avenues, in order to illustrate various aspects of problem solving.

Think of the block as an agent (or if you prefer, think of yourself as the block) and consider the forces on it. There is its weight W vertically downward, the reaction of the surface normal to the plane R and the friction F_r along the plane, and finally, the force F pulling the block. Once again we represent the direction of a force by an arrow in that direction and the magnitude of the force by the length of the arrow.

We shall do a simpler example first by considering the case $\theta = 0$ when the force F is in a direction parallel to the plane. We could balance forces horizontally and vertically, which would involve three forces in each case. It looks simpler to balance forces parallel to the plane and normal to the plane. Of course, both methods (and any other resolution) lead to the same result.

Resolving parallel to the plane in Figure 2.16 (with $\theta = 0$), we have

$$F = F_r + W \sin \alpha,$$

where $0 \leq \alpha < \pi/2$ is the slope of the plane. Resolving normal to the plane:

$$R = W \cos \alpha.$$

At the point of slipping, $F_r = \mu R = \mu W \cos \alpha$ so

$$F = \mu W \cos \alpha + W \sin \alpha. \qquad (2.13)$$

The stationary (maximum or minimum) value of F is obtained by setting $\mathrm{d}F/\mathrm{d}\alpha = 0$. Differentiation of (2.13) with respect to α shows that the derivative is zero for $\alpha = \cot^{-1} \mu$. For this α, $\cos \alpha = \mu \big/ \left(1 + \mu^2\right)^{\frac{1}{2}}$ and $\sin \alpha = 1 \big/ \left(1 + \mu^2\right)^{\frac{1}{2}}$. So F has the value

$$F = \left(1 + \mu^2\right)^{\frac{1}{2}} W.$$

For α = 0, we have $F=\mu W \leq \left(1+\mu^2\right)^{\frac{1}{2}} W$, so $a = \cot^{-1}\mu$ must be a maximum, not a minimum. Assuming $\mu < 1$, the smallest value for F is μW at $\alpha = 0$ (i.e., no slope at all).

So following the algebra, we seem to have come to the conclusion that the slope only makes things worse in terms of minimizing the force! However, if we take $\alpha = 0$, we do not raise the block at all. The practical answer, other things being equal, seems to be that we want the gentlest slope possible. Of course, this is obvious! The more we wish to raise the block vertically, the more we have to pull! So we learn two things: first, to think about what outcome we expect before doing a calculation and second that we need to interrogate the mathematics, not be led blindly by it. Here we needed to check whether the stationary value for the force is a minimum and to not be misled by the hope that that is indeed what we would find.

A more relevant calculation might be to determine the angle required for a given force (which we would choose as the maximum the workers could exert). We have to solve Equation (2.13) for α. The result is not very illuminating, but the method is, so we shall present it here. We isolate either the sin or the cos term on one side of the equation and square. Isolating the sin term on the left-hand side and squaring gives

$$\sin^2 a = \left(\frac{F}{W} - \mu\cos a\right)^2.$$

We can now substitute for $\sin^2\alpha$ in terms of $\cos^2\alpha$, giving an equation involving only cosine terms: this was the point of the initial manipulation. Using $\sin^2\alpha = 1 - \cos^2\alpha$, we get

$$(\mu^2+1)\cos^2 a - \frac{2\mu F}{W}\cos a + \left(\frac{F^2}{W^2} - 1\right) = 0.$$

This operation has yielded a quadratic for $\cos\alpha$ (with no $\sin\alpha$ terms) with the solution

$$\cos a = \frac{\mu\beta \pm \sqrt{\mu^2+1-\beta^2}}{\mu^2+1}, \qquad (2.14)$$

where $\beta = F/W$. Now we want $\beta < 1$, that is, $F < W$, or else there is no point in dragging the block up a slope. If we take the negative square root with $\beta < 1$, this gives $\cos\alpha < 0$ or $\alpha > \pi/2$ which is obviously not possible. Thus, we take the positive root.

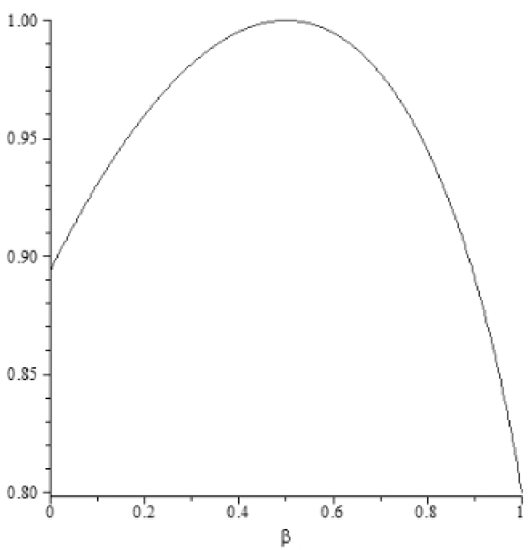

Figure 2.17: The function $\cos\alpha$ plotted against $\beta = F/W$ from Equation (2.14) for $\mu = 0.5$

Finally, look again at the expression, we have obtained in Equation (2.14). For $\beta = 0$, we have $\cos\alpha = \left(1+\mu^2\right)^{-1/2}$. Does this mean that is we apply a zero force ($F = \beta W = 0$) the block will levitate up the slope? Clearly not! So what is wrong? As a clue consider that if $F = 0$, the block will, if anything, slide *down* the plane. So our solution does not apply to the problem that we set for small values of F or β. It is useful to draw a graph. Figure 2.17 shows $\cos\alpha$ plotted against β for a value of $\mu = 0.5$. At the peak of the curve, $\cos\alpha = 1$ or $\alpha = 0$. This is therefore the point at which the force is sufficient to move the block on a flat plane. Values to the left of this are not solutions of the problem; only for values of β to the right of the maximum do we get a valid slope.

2.15 PULLING AT AN ANGLE ON A FLAT PLANE

We can now investigate how much difference pulling at an angle up an inclined plane would make. However, before we do this, consider pulling at an angle $\theta \neq 0$ on a flat plane (α = 0). Before we do any calculation, let us see why we expect an optimum angle. If the angle θ in Figure 2.18 is negative, the block is pulled onto the plane so the reaction force is bigger than it might otherwise be and friction is increased. If the angle θ is too large and positive, much of the effort is going into lifting the block off of the plane, and while this reduces friction, the component of force pulling the block along the plane is diminished. Somewhere in between, we expect to find the choice of θ that minimizes the force required.

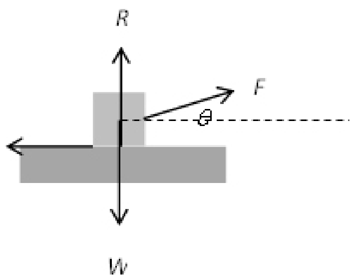

Figure 2.18: Pulling a block at an angle

Resolving forces horizontally in Figure 2.18, we have

$$F_r = F\cos\theta$$

and resolving vertically,

$$W = R + F\sin\theta.$$

The block will slide once $F_r = \mu R$, so using this and eliminating R gives

$$\mu(W - F\sin\theta) = F\cos\theta, \tag{2.15}$$

or

$$F = \frac{\mu W}{\cos\theta + \mu\sin\theta}. \tag{2.16}$$

This gives the force required for a given value of the angle θ. For the minimum force, we differentiate (2.15) (or 2.16) with respect to θ and set $dF/d\theta = 0$ to get

$$\theta = \tan^{-1}\mu.$$

This is reasonable: for negligible friction ($\mu << 1$), θ is small: there is no point in pulling anything other than close to horizontally. For the minimum force, we find from (2.16)

$$F = \frac{\mu W}{\sqrt{1+\mu^2}}. \qquad (2.17)$$

This also tells us that if μ is small, there is not much point in adjusting the angle: for $\theta = 0$, $F = \mu W$ so adjusting the angle has reduced the force in Equation (2.17) only by terms of order μ^2. We would be far better off reducing μ because the force required depends linearly on μ. So reducing μ by a factor 2 would roughly halve the force required.

At the other extreme, for large μ, consider the case that μ has its maximum value of $\mu = 1$. Then pulling at 45° (i.e., $\theta = \tan^{-1}1$) reduces the required force in (2.17) by a factor of $\sqrt{2}$, and this is the best we can do by adjusting the angle. Incidentally, in this case, we really do have a minimum and not a maximum: for example, if $\theta = 0$, we have

$$F = \mu W > \frac{\mu W}{\sqrt{1+\mu^2}},$$

that is, for angles away from the stationary point, the force required is larger, hence, the stationary point is a minimum.

Alternatively, we can solve (2.15) for θ for a given force. The result is again not very illuminating, but the method is. We could follow the approach we used to solve (2.13), but there is a more interesting way. Compare (2.13):

$$F = \mu W \cos\alpha + W \sin\alpha.$$

and (2.15), rewritten as

$$W = \frac{F}{\mu}\cos\theta + F\sin\theta.$$

We see that the transformations $W \leftrightarrow F$ (or $\beta \to 1/\beta$), $\mu \to 1/\mu$ and $\theta \leftrightarrow \alpha$ converts one equation into the other. Therefore, the solution of the second equation is found by applying this transformation to the solution, Equation (2.14), of the first equation: from Equation (2.14), we get

$$\cos\theta = \frac{\mu^{-1}\beta^{-1} + \sqrt{\mu^{-2} + 1 - \beta^{-2}}}{\mu^{-2} + 1}$$

and tidying up:

$$\cos\theta = \frac{\mu}{\beta}\left(\frac{1 + \sqrt{\beta^2 + \beta^2\mu^2 - \mu^2}}{1 + \mu^2}\right).$$

This idea of transforming one problem (that we have solved) into another (which we wish to solve), and thereby obtaining the solution by transformation, is exploited in various contexts in theoretical physics.

2.16 PULLING AT AN ANGLE ON AN INCLINED PLANE

Now to the calculation for the inclined plane: Figure (2.18) shows the forces forming a closed figure, which will be the case up until the block is about to slip. We could once again resolve forces horizontally and vertically, but this will involve three forces in each case and some trigonometric functions of $\alpha + \theta$. It seems as if it might be easier to consider the balance of forces parallel to the plane and perpendicular to it.

Resolving parallel to the plane:

$$F\cos\theta = F_r + W\sin\alpha. \tag{2.18}$$

Resolving perpendicular to the plane:

$$W\cos\alpha = R + F\sin\theta. \tag{2.19}$$

Once more, if the block is just about to slip, the frictional force is directly related to the normal reaction through the coefficient of friction for the two surfaces, μ by

$$F_r = \mu R. \tag{2.20}$$

We can in principle solve these equations for the angle θ. We start by combining Equations (2.18) to get F as a function of θ for fixed W and μ giving

$$F\cos\theta = \mu(W\cos\alpha - F\sin\theta) + W\sin\alpha. \tag{2.21}$$

We can find the angle that minimizes F by putting $\frac{dF}{d\theta}=0$. Differentiating (2.21) gives

$$\cos\theta\frac{dF}{d\theta} - \sin\theta F = -\mu\frac{dF}{d\theta}\sin\theta - \mu F\cos\theta,$$

or, when we put $dF/d\theta = 0$,

$$\theta = \theta_m = \tan^{-1}\mu$$

for the value of θ at the minimum. Is this answer reasonable? If μ is small, then θ_m is small and we should pull almost parallel to the plane to minimize the effort required to move the block: we cannot reduce by much the already small friction. As μ increases, we need a larger angle to the plane. The angle θ_m is independent of the slope α of the ramp, which is perhaps a bit unexpected. But what force do we have to exert? We have assumed this can be adjusted at will. Let us substitute θ_m back into Equation (2.21) and see. This gives

$$F\left(\frac{1}{\sqrt{1+\mu^2}} + \frac{\mu^2}{\sqrt{1+\mu^2}}\right) = W\left(\mu\cos\alpha + \sin\alpha\right),$$

or

$$F = \frac{1}{\sqrt{1+\mu^2}}W\left(\mu\cos\alpha + \sin\alpha\right). \tag{2.22}$$

The maximum value of $\mu\cos\alpha + \sin\alpha$ is $\sqrt{(1+\mu^2)}$ when $\alpha = \cot^{-1}\mu$. Then $F = W$ and $\theta + \alpha = \pi/2$. In other words, the ramp is of no benefit: we may as well lift the weight vertically!

So a more realistic calculation might be to investigate the angle of the ramp given a fixed force. We have just found that the best angle θ is independent of the ramp angle α, so we put $\theta = \tan^{-1}\mu$ and solve (2.22) for α. But apart from the factor $\sqrt{(1+\mu^2)}$, this is the same as (2.13). So from the solution to (2.13), Equation (2.14), we have

$$\cos\alpha = \frac{\mu\beta + \sqrt{\mu^2 + 1 - \beta^2}}{\mu^2 + 1}.$$

where now

$$\beta = \sqrt{\mu^2 + 1}\,\frac{F}{W}.$$

2.17 SOLUTION OF PROBLEM 2

Let us put in some values. A stone block in the pyramid has a weight of about 60,000 N. Let us assume it is being moved on rollers, so friction might be $\mu = 0.1$ (say). A man can pull rather less than his body weight, so say 300 N. Suppose we employ 50 men to move the block so $\beta \sim F/W \sim 0.25$. From (2.14), we get $\alpha \sim 7°$. The length of the ramp to the height of the pyramid must therefore be cot 7° ~ 9. So the length of the ramp should be ~10 times its height.

Unfortunately, this estimate depends sensitively on the value of β and μ. We should therefore add some error bars. We have, fixing μ and varying β,

$$\delta\cos a = -\sin a\,\delta a = \frac{\mu\delta\beta - \beta\delta\beta\left(\mu^2 + 1 - \beta^2\right)^{-\frac{1}{2}}}{\mu^2 + 1} \sim \delta\beta(\mu - \beta),$$

where we have taken $\mu^2 << 1$ and $\beta^2 << 1$ for our assumed values of μ and β. So $\delta a \sim (0.15/\sin 7°)\delta\beta \sim 1.2\delta\beta$. In degrees, this gives $\delta a \sim 70°\delta\beta$. So we need to know β to 1% to get α to a factor of 2 (since $\delta\beta = 0.01$ gives $\delta\alpha \sim 7°$, comparable with the value we found for α). A similar conclusion holds for μ. It is worthwhile remembering that this problem is generic: since a function varies slowly near a stationary point, locating a maximum or minimum accurately requires accurate data!

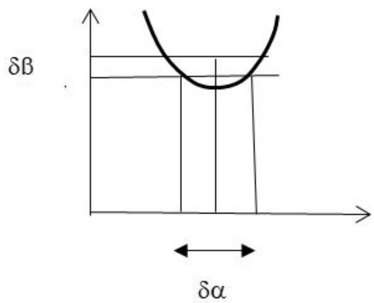

Figure 2.19: Near a minimum, a small change in β gives an inaccurate estimate of α

2.18 TIPPING POINT

Where along the side of the block should we attach the rope? Intuitively, we know that it should be at the lowest point, since this not only stops the block from tipping but also helps to lift it off of the ground and minimize the friction. How can we show this? We do an easier problem first.

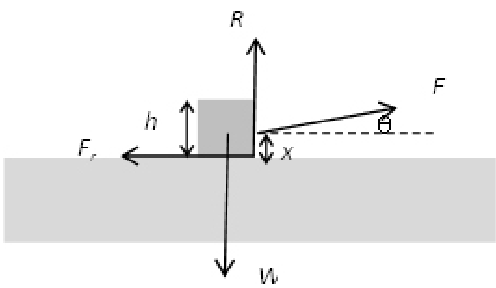

Figure 2.20: Considering the block as an extended body, the position at which the rpe is attached will determine if it tips or slides

In Figure 2.20, we consider that a square block is just at the point of tipping when it starts to slide. How do we know this is a tipping point? Consider approaching the situation by increasing the force gradually while the block refuses to slide. As the force increases, the far end of the block will begin to leave the ground. At this point, the reaction force must be through the only point of contact with the ground, which must also be where friction is acting.

Why do we draw the figure like this and not with the block tipping over (Figure 2.21)? It would be perfectly possible to analyze the situation mathematically with the block at an arbitrary angle, but toppling from this position is more likely than toppling from the horizontal. We can see this because once the block starts to tip off of the ground, the line of action of the weight moves toward the pivot end. So the weight exerts less of a restoring moment as the block tips. In other words, if the block is going to slip without tipping, it had better do so by the time the block is about to pivot off the ground or it will not do so at all. So this is a situation where a little intuition can save us some complication in the mathematics.

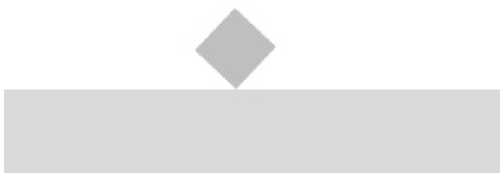

Figure 2.21: A block at tipping point

To solve the problem, we consider the block to be at the point of sliding and resolve horizontally in Figure 2.20:

$$F\cos\theta = F_r = \mu R, \tag{2.23}$$

and vertically:

$$W = R + F\sin\theta = \frac{F\cos\theta}{\mu}(1+\mu\tan\theta), \tag{2.24}$$

where the final expression arises by substituting for R from Equation (2.23). We now also take moments about the point of contact in Figure 2.20. Of course, in reality, rather than in the two-dimensional projection of the figure, this is a line, not a point, but nevertheless, we usually talk about taking moments about a point! We choose this point because the unknown quantities F_r and R have zero moment about this point. They are therefore automatically eliminated from the resulting equation, thereby simplifying the resulting algebra.

We know that taking moments about any other point would lead to the same result eventually. The moment equation about the line of contact is

$$\frac{h}{2}W = xF\cos\theta. \tag{2.25}$$

We now solve these Equations (2.24) and (2.25) for x/h in terms of θ by eliminating W/F. A little algebra gives

$$\frac{x}{h} = \frac{1}{2}(\tan\theta + \frac{1}{\mu}). \tag{2.26}$$

This is the condition for the block to slide just at the point where it is about to topple. To find the condition that it slides before toppling, we could rework the equations with $F_r > \mu R$ at the point of toppling (because the friction in this case would have to be larger than it actually is to prevent slipping). More simply, we can see that if the point of application is lower than implied by (18.4), the couple on the block will be reduced and it is less likely to topple. Therefore, the block will slide before toppling if

$$\frac{x}{h} \le \frac{1}{2}(\tan\theta + \frac{1}{\mu}).$$

We look at some special cases to check the answer is reasonable and to draw some conclusions. Consider first the case $\theta = 0$. Then $x/h \le 1/(2\mu)$. Thus, for $\mu \le 1/2$, it does not matter where we attach the rope: the block will always slide before toppling. For $\mu = 1$, we get $x \le h/2$ and we must attach the rope at or below the half-way point. In this situation, we can see that the block cannot tip because there is no clockwise moment; so this is in agreement with our result.

If we now choose the angle θ for the minimum force, $\tan\theta = \mu$ (from Section 2.15), we get

$$\frac{x}{h} \le \frac{\mu^2 + 1}{2\mu}.$$

The right-hand side is always >1. To see this, consider $\frac{\mu^2+1}{2\mu} - 1 = \frac{(\mu-1)^2}{2\mu} > 0$. Thus, if we choose the angle to correspond to the minimum force, the block will always slide before tipping. This is something we might expect the ancient Egyptians to have discovered by trial and error in the course of laying many thousands of blocks.

2.19 TIPPING ON AN INCLINED PLANE

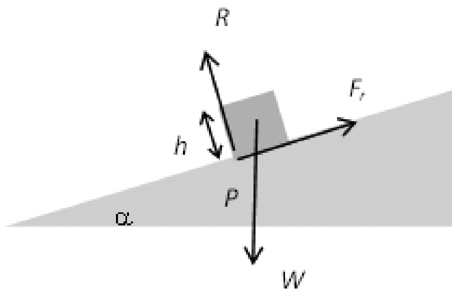

Figure 2.22: Block on an inclined plane

A slightly different problem occurs for a block on an inclined plane (Figure 2.22). Once again, we consider the simpler case of a cubical block of side h, first for a block that is not being pulled. The block will now tip, if it tips at all, about the lower edge so if the block is at the point of sliding as it is about to topple the reaction force will act through the lower edge. To look at the condition on the slope of the plane, once again we could resolve the forces parallel to and normal to the plane and take moments about a convenient point, in this case, to get the equation with the fewest forces, about the lower edge, P. Since the block is sliding down the slope, the frictional force acts up the plane.

However, looking at the diagram (Figure 2.22), we see that W produces an unbalanced couple about the lower edge at P. Thus, at the point of toppling, W must also act through the edge P. So if toppling occurs at the point of slipping, $\alpha = 45^o$. If $\alpha < 45^o$, the block slip before it topples.

Resolving normal to the plane

$$R = W\cos\alpha. \tag{2.27}$$

Resolving parallel to the plane

$$W \sin\alpha = F_r \leq \mu R = \mu W \cos\alpha,$$

using (2.27) in the final step. Thus, the block will topple before slipping if μ. With α = 45°, this gives μ = 1, that is, the block will slip before it topples if μ < 1.

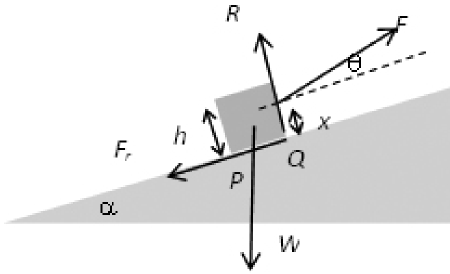

Figure 2.23: The block tips or slides depending on the point of application and the angle of the rope

Finally, we look at a block pulled up an inclined plane (Figure 2.23). Our experience shows us not to plunge straight into the equations. So what do we expect? As we have drawn the figure, the moment of F about P is clockwise, hence acts to tip the block about the leading edge at Q. If the angle θ is increased sufficiently, or the distance x is decreased, then the moment of F about P will act to tip the block instead about the trailing edge at P. The condition for the situation in Figure 2.23 is

$$\tan\theta < \frac{x}{h}.$$

Assume again that the block is on the point of tipping when it starts to slide. Resolving along the plane:

$$W \sin\alpha + F_r = F \cos\theta,$$

and normal to the plane:

$$F \sin\theta + R = W \cos\alpha.$$

Again, taking moments about the leading edge:

$$xF\cos\theta = \frac{Wh\cos\left(\frac{\pi}{4}-a\right)}{\sqrt{2}}.$$

Putting tan $\theta = \mu$ for the minimum force and solving for x, we get

$$\frac{x}{h}=\frac{1+\mu^2}{2\mu}\left\{\frac{(\cos a+\sin a)}{\left[\cos a+\sin a+\left(\frac{1}{\mu}-1\right)\sin a\right]}\right\}.$$

In this equation, we have written the denominator in such a way that it is obvious that for $\mu < 1$, the trigonometric factor in curly brackets is <1 because the denominator is greater than the numerator. Thus, on an inclined plane, we must attach the rope nearer the base; that is, we appear to have shown that the slope makes it more likely that the block will topple before sliding! How can we understand this?

There are two factors at work. On the one hand, the moment of the weight about the leading edge increases as the slope gets steeper acting as a stabilizing influence against tipping. On the other hand, the reaction on the ground is less because the component of the weight normal to the slope is the reduced. Thus, we require a larger applied force before the friction reaches its limiting value, making the block more prone to tipping. Only the calculation can tell us which of these are the dominant effect. The pyramid builders will not have done any of these calculations (although they were pretty nifty with trigonometry) so we have not learnt much about the Ancient Egyptians; what we have learnt is something of how to construct and interrogate mathematical models.

2.20 LEVERS

Consider now how blocks might be levered into place. The figure shows two arrangements: where is the fulcrum best placed?

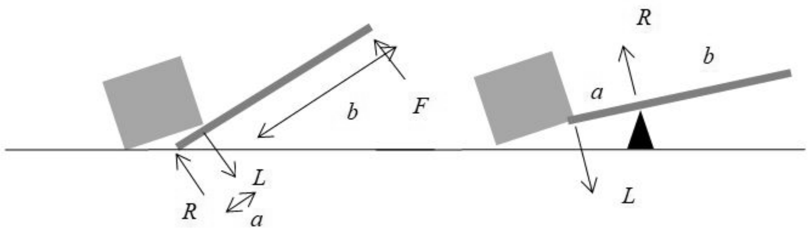

Figure 2.24: Various ways of applying a lever

Figure 2.24 shows the forces on the lever: the applied force, F, the load, L, and the reaction from the fulcrum, R. Note how we have drawn the forces in the figure, namely normal to the lever. The forces will have components parallel to the lever, and these must balance, but they will not help lift the weight; thus, the equilibrium of the parallel components is irrelevant to the problem so we need not consider it.

Since we are interested in the relationship between the applied force and the load we take moments about the fulcrum. In the first case, we get

$$aL = (a + b)F,$$

and therefore,

$$F = \frac{aL}{a + b}.$$

In the second case, we have

$$aL = bF,$$

so

$$F = \frac{aL}{b}.$$

The applied force is clearly smaller in the first case, because the lever arm (the distance from the applied force to the fulcrum) is longer in this case.

2.21 STRESS AND STRAIN

Finally, we consider how the blocks may have been pulled up a ramp by ropes. Over a limited range of the force F applied to it, a rope obeys *Hooke's law*:

$$F = kx,$$

where x is the extension (the change in length) and k is a constant which will depend on the material and dimensions of the rope. Since we do not have a sample of Egyptian rope on which to evaluate k the equation in this form is not very useful. At a more fundamental level, the constant k is related to *Young's modulus* Y of the material:

$$k = \frac{YA}{l}.$$

This enables us to express the relation in a more fundamental way: for a rope of length l and cross-sectional area A

$$\frac{F}{A} = Y\frac{x}{l}.$$

The quantity F/A (force per unit area) is defined as the stress applied to the material, and x/l (extension divided by original length) is the strain, so Hooke's law can be stated as stress $\propto$ strain, with Young's modulus as the constant of proportionality.

At large stresses, the material will deviate from this law: beyond the so-called elastic limit the material will not return to its original state when the stress is removed, and eventually of course the rope will break.

Example: As an example, the ancient Egyptians invented the method of making ropes by twisting fibers together. The breaking stress of hemp which the pyramid builders would have plaited into ropes, might have been around 10^8 N m^{-2}. To exert a force of say 5×10^4 N on a 50,000 N block of stone would require rope with a cross-section of $5 \times 10^4/10^8 = 5 \times 10^{-4}$ m^2, so about five ropes of 6 mm radius.

2.22 CHAPTER SUMMARY

- Force is a vector quantity with magnitude and direction and point of application
- Newton's Third Law: Between any pair of agents, to every action, there is an equal and opposite reaction
- A body is in static equilibrium if the net force and the net couple on it (about any fixed point) are zero; in particular the components of force must balance in every direction
- The maximum static frictional force on a body is given by the coefficient of friction μ times the reaction force R and is independent of area
- The center of gravity of a body is the point about which the moment of its weight is zero
- To solve a problem in mechanics, we

  i) draw a diagram

  ii) identify the agents and their interactions on the diagram

  iii) think what answer you expect

  iv) express the problem mathematically by considering the forces on each agent

  v) express the solution analytically in symbols (if possible)

  vi) check the dimensions are correct

  vii) check the answer looks reasonable for special cases

  viii) express the solution graphically if possible

  ix) express the solution in words and decide if the solution meets our expectations; if not investigate why not

  x) substitute numerical values if required in consistent units (and check that the order of magnitude of the outcome is reasonable).

2.23 EXERCISES

1. Figure 2.25 shows (schematically) a bulldozer of mass M pushing a smaller block of mass m. Add the missing forces due to friction, labeling them F_9 on the bulldozer and F_{10} on the block.

What are the relations (greater than, equal to, or less than) between the following forces:

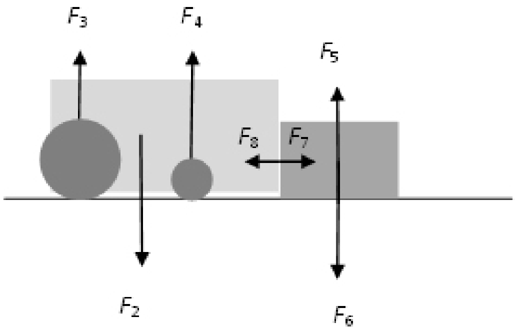

Figure 2.25: Bulldozer and block, question 1

a) F_7 and F_8

b) F_9 and F_8

c) F_3, F_4, and F_2

d) F_5 and F_{10}?

What is the relation between F_2 and the mass of the bulldozer?

What is the acceleration of the system in terms of the masses of the bulldozer and block and the forces acting on the system?

2. The diagram (Figure 2.26) shows a model of a nutcracker consisting of two equal hinged levers of length l pushing on a spherical nut of radius a. The coefficient of friction between the nut and the levers is μ. A force F is applied at right angles at the end of each of the levers.

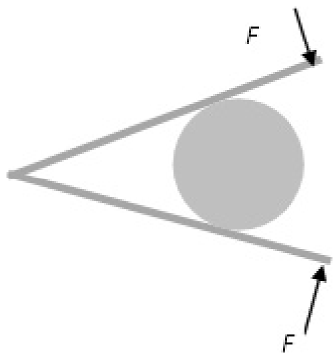

Figure 2.26: A model nutcracker and nut (question 2)

Draw a diagram of the forces acting on the nut and of the forces on the levers.

If the contact between the nut and the lever is a distance x along the lever, by taking moments about the hinge show that the normal reaction on the nut is Fl/x.

By considering the horizontal forces acting on the nut, show that the nut will start to slip when $x = a/\mu$ at which point the normal force on the nut is $\mu lF/a$. Deduce that, other things being equal, it is easier to crack a smaller nut.

3. Point weights W_1 and W_2 are separated by a distance d.

a) Show that their center of gravity is a distance

$$\frac{W_2 d}{W_1 + W_2}$$

from W_1 and find its distance from W_2.

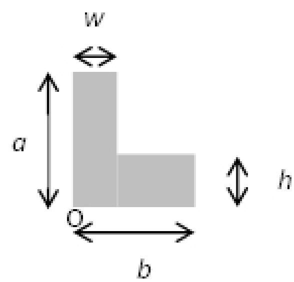

Figure 2.27: An L-shaped figure (question 3)

b) Find the center of gravity of the uniform L-shaped figure shown in Figure 2.27 if $a = b = 10$ cm and $h = w = 2$ cm.

c) Describe the orientation of the object if it is hung from the corner at O.

4. A typical femur has a Young's modulus of 17,900 N mm^{-2}, a length of 500 mm and a cross-sectional area of 330 mm^2. How much shorter is your femur when you are you standing up?

5. Figure 2.2 shows a horizontal rod fixed at A but free to expand to the right. The spring constant is 10 Nm^{-1} (i.e., a force of 10 N produces a contraction of 1 m) and the coefficient of expansion of the rod (i.e., the fractional increase in length per degree) is 2×10^{-5} m per degree. The length of the rod is 1 m supported on a knife edge at its center. A mass of 0.1 kg is hung from the right end.

a) The system is initially in equilibrium with the rod horizontal. Show that the force exerted on the rod by the spring is 0.1g. In which direction does it act?

b) With the position of the knife-edge fixed with respect to A, the rod is now heated by 1°C. Neglecting the mass of the rod, through what angle does it move?

Is this the basis of a useful thermometer?

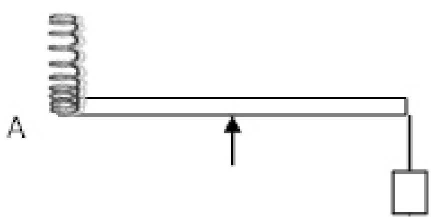

Figure 2.28: Question 5

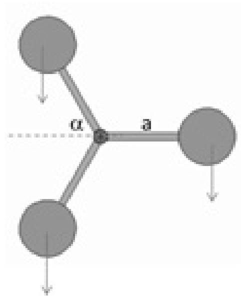

Figure 2.29: Question 6

Figure 2.29 shows a proposed perpetual motion machine. It is argued that the force on the two balls on the left if greater than the force on the ball on the right, so the machine will turn forever. What is the fallacy?

CHAPTER 3

Kinematics

Problem 1: Motion in one dimension

"According to the court report he was traveling at 60 mph. Had he been obeying the 30 mph speed limit he would have stopped in half the distance." What is wrong? What should it say?

3.1 CONSTANT SPEED

Let us start with one of the iconic pictures of kinematics first proposed by the 14th century French philosopher Nicole Oresme. This is a graph of the speed of a body moving at constant speed against time (Figure 3.1). Since

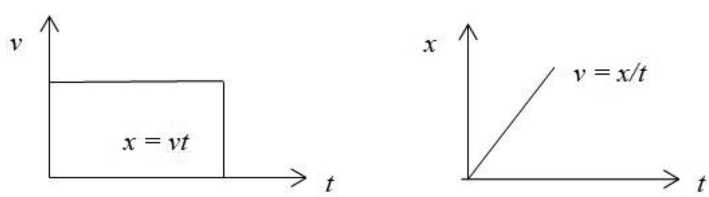

Figure 3.1: Constant speed in one-dimension

by definition, v = distance/time = x/t, we have $x = vt$: that is, the distance traveled is the area under the graph. Alternatively, we plot distance against time. The speed is then the slope of the graph. This graphical representation resolved a long-standing puzzle. The puzzle was this: what does it mean to divide unlike quantities, that is, quantities with different units? We certainly cannot add or subtract them; dividing like quantities is valid, of course, since it gives a pure number. Oresme's graphs elucidate the meaning of speed as distance divided by time.

Suppose we transform to a frame of reference x' moving with constant speed u in the positive x-direction. Then the x coordinate transforms as

$$x' = x - ut. \tag{3.1}$$

So

$$v' = \frac{x'}{t} = \frac{x}{t} - u = v - u. \tag{3.2}$$

The transformation law (3.1) between coordinates is called a Galilean transformation and Equation (3.2) is the addition law for velocities in one dimension.

3.2 CONSTANT ACCELERATION

If a body moves under constant acceleration, we can draw similar graphs to those for constant speed. Then from the area under the graph of acceleration against time, we have $v = at$. Similarly, the acceleration is the slope of the graph of v against t. We can go one step further: the distance traveled is the area under the v–t graph so, assuming the body starts from rest, $x = ½\, at^2$ (the area of a triangle is ½ × base × height). If the body starts with a speed u, there is and additional area ut to add on so

$$x = ut + \frac{1}{2}at^2. \tag{3.3}$$

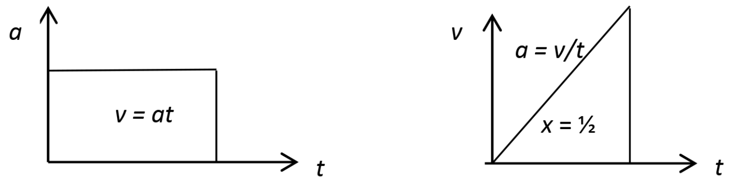

Figure 3.2: Constant acceleration

For a body starting from rest, we have

$$x = \frac{1}{2}at^2 = \frac{v^2}{2a},$$

or

$$v^2 = 2ax,$$

which gives us a relation between speed and distance rather than speed and time.

You might guess that starting from a speed u, this equation becomes

$$(v-u)^2 = 2ax.$$

It is instructive to see why this must be wrong. If we reverse the sign of acceleration a (so the body is slowing down from its initial speed), then the right-hand side changes sign, but the left-hand side (being a square) does not. Therefore, this cannot be a valid formula.

We can obtain the correct result starting from (3.3) and putting $t = (v-u)/a$; after some algebra, we arrive at

$$v^2 = u^2 + 2ax. \tag{3.4}$$

But it is more instructive to see how this arises from the Galilean transformation. Let the motion be viewed from a frame moving at constant speed u along the positive x axis. Then, in this frame, $x' = x - ut$; and $v' = v - u$. Thus

$$v^2 = 2ax = 2a(x' + ut) = 2ax' + \frac{2auv}{a},$$

or

$$v^2 - 2uv = 2ax',$$

and so

$$(v-u)^2 - u^2 = v'^2 - u^2 = 2ax',$$

which is the result (in the primed frame) we were seeking to prove.

Finally, the most general approach is to use calculus. For a body undergoing constant acceleration a from an initial speed u, we have

$$\frac{d^2x}{dt^2} = a. \tag{3.5}$$

Integrating gives

$$\frac{dx}{dt} = u + at,$$

where u is a constant or

$$v = u + at. \tag{3.6}$$

Thus, u is the speed at $t = 0$. Integrating again, we get

$$x = ut + \frac{1}{2}at^2, \tag{3.7}$$

assuming that the body starts from $x = 0$ at $t = 0$. Given the acceleration, the motion of the body is completely determined once we specify the initial speed and position. Thus, Equations (3.6) and (3.7) contain all the information about the motion, and we have seen that we can manipulate them to obtain the speed–distance relation of Equation (3.4). However, it is sometimes useful to know that we can obtain (3.4) directly by integration in the following way. To obtain a speed–distance relation, we convert (3.5) from distance and time to distance and speed, we have

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{ds}\left(\frac{ds}{dt}\right) = \frac{dv}{ds}v,$$

where the third equality is obtained from the chain rule of calculus. If a is a constant, this integrates immediately to (3.4).

The advantage of this method is that it can be applied whatever the law of motion. So, for example, if the need arose, we could use this method to solve for the motion of a body with a constant rate of change of acceleration.

3.3 A BODY PROJECTED VERTICALLY UNDER GRAVITY

Let us take as an example a body projected vertically from height $y = 0$ with speed v subject to a deceleration $-g$. We expect the body to reach a maximum height at which point its speed will be zero. It will then return to the origin with acceleration g. The two parts of the motion are symmetrical – the speed on the way down is the same as that on the way up at corresponding points. If this were not the case, we could extract the additional energy on the way up or down and create a perpetual motion machine. So the interesting issues are the maximum height and the time taken. The motion is governed by

$$y = vt - \frac{1}{2}gt^2 \tag{3.8}$$

on the way up. You might think that this equation needs to be modified for the downward leg, since v is now negative. In fact, the symmetry of the two parts ensures that the same equation holds for each leg. To be convincing, we shall show this explicitly in a moment. First let us calculate the maximum height.

The simplest approach is to complete the square:

$$y = -\frac{1}{2}g\left(t - \frac{v}{g}\right)^2 + \frac{v^2}{2g}. \qquad (3.9)$$

The maximum occurs when the negative term is as small as possible, namely zero. So $y_{max} = h = \frac{v^2}{2g}$ at $t = \frac{v}{g}$. Of course, the same result can be obtained by differentiation to find the maximum. This result is valid whatever the downward motion, so we can use it to set up the equation for the fall.

The body starts at $y = \frac{v^2}{2g}$ at time $t = \frac{v}{g}$ with speed 0. The acceleration is still in the $-y$ direction, so is still $-g$. Thus, using $y = y_0 + ut + \frac{1}{2}at^2$, we have

$$y = \frac{v^2}{2g} + 0 - \frac{1}{2}g\left(t - \frac{v}{g}\right)^2,$$

which is the same as (3.8). Rearranging, we get, for $t > v/g$,

$$y = vt - \frac{1}{2}gt^2,$$

which is, naturally, the same as (3.8). So we do not have to consider the up and down motion separately.

Problem 1: We can now tackle the problem. It is often useful to begin a problem by creating a visual representation, in this case a graph. Since the problem is dealing with speed and distance, we use these as axes.

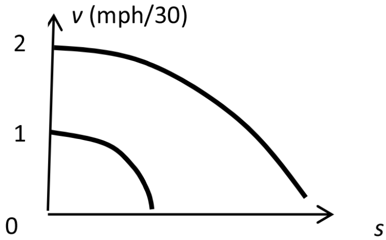

Figure 3.3: Stopping distance, problem 1

A convenient unit here is to measure speed in units of 30 mph. In the absence of any other information, we assume constant deceleration. Then stopping distance s_2 from a speed of 2 units with deceleration a is given by

$$4 = 2as_2.$$

The distance s_1 in stopping from 1 speed unit is given by

$$1 = 2as_1,$$

from which it is clear that $\frac{s_1}{s_2} = \frac{1}{4}$ and not $\frac{1}{2}$. We see that the stopping distance increases as the square of the speed under constant deceleration. One might argue that constant dissipation of energy would be a better assumption. We shall return to this once we have formally introduced the concept of energy.

Problem 2 (Motion in two dimensions):

One of the first military applications of mathematics (certainly one of the first for financial gain) was Galileo's application of projectile motion to aiming a canon. Our problem in the following sections is a slight variation on the theme: Given the height of a castle wall what is the angle of release of a projectile from the top of the wall that gives it maximum range?

3.4 MOTION IN TWO DIMENSIONS

Another of Galileo's many achievements was the recognition that the orthogonal components of motion in two dimensions are independent: we can treat vertical and horizontal motion separately and combine the results. This is an empirical result; in relativity theory, it is not true. Thus, various combinations arise: constant speed in both directions; constant speed in one and constant acceleration in the other; constant acceleration in both directions. We treat each in turn.

3.5 ADDITION OF VELOCITIES

Since the orthogonal components of velocity are independent, we can add them separately. Thus, if

$$\boldsymbol{u} = u_1\mathbf{i} + u_2\mathbf{j},$$

and

$$\boldsymbol{v} = v_1\mathbf{i} + v_2\mathbf{j},$$

then

$$\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1)\mathbf{i} + (u_2 + v_2)\mathbf{j}.$$

This is equivalent to the parallelogram law, as shown in Figure 3.4.

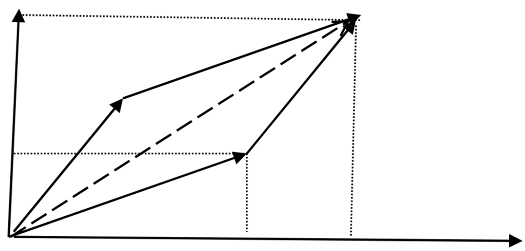

Figure 3.4: The addition of two vectors by components is equivalent to the parallelogram law

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Example: A missile appears to be approaching you from 30° north of west. If you are flying north at the same speed as the missile, what is the true direction of travel of the missile?

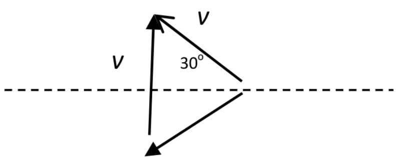

Figure 3.5: Example triangle of velocities of relative motion

From the isosceles triangle in the figure, the true direction is 30° south of east.

3.6 PROJECTILE MOTION

In this section, we add motion at constant speed in the horizontal direction to motion at constant acceleration vertically downward. This describes, for example, the motion of a projectile in the absence of air resistance, which is what we need to solve problem 2. Let's start with some simpler examples.

Example 1: A projectile launched from a tower

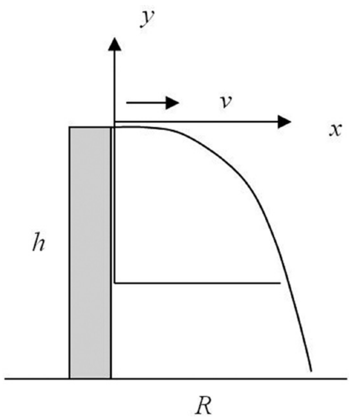

Figure 3.6: Launch of a projectile horizontally from a tower

Suppose we launch a projectile horizontally with speed v from a height h. Where does the projectile land? This problem nicely illustrates the independence of the horizontal and vertical motion. We begin by sketching a diagram and deciding (carefully) the direction of our axes and the origin. Suppose y is measured vertically *upward* and x horizontally with the origin at the *top* of the tower. For the horizontal motion at constant speed, we have

$$x = vt, \tag{3.10}$$

and for the vertical motion at constant acceleration from rest

$$y = -gt^2. \tag{3.11}$$

The projectile hits the ground when $y = -h$, hence when $-h = -gt^2$ or $t = \sqrt{\frac{h}{g}}$. Thus, the range is

$$R = v\sqrt{\frac{h}{g}}.$$

First let us check that this is dimensionally correct: $[R] = LT^{-1}\left(\frac{L}{LT^{-2}}\right)^{\frac{1}{2}} = L,$ which is right. And obviously, the range increases with increasing v and h, and we would also expect the projectile to go further if the acceleration due to gravity was weaker. Note that we could not have guessed the result by dimensional analysis alone, because there are two length scales in the problem: h and $\frac{v^2}{g}$. The range is the geometric mean of these:

$$R = \left(\frac{v^2}{g}h\right)^{\frac{1}{2}}.$$

Finally, we derive the form of the trajectory. Equations (3.10) and (3.11) give the curve in parametric form (with t as the parameter). We eliminate t to get the coordinate form:

$$y = -\frac{g}{v^2}x^2,$$

which is an inverted parabola through the origin.

Example 2: The range of a projectile on a flat surface

Consider next a projectile launched from the point $x = 0$, $y = 0$ at an angle θ to the vertical with speed v. First think about what we expect. The projectile reaches a highest point and falls back again. In the absence of air resistance, the motion about the highest point is symmetrical. We can see this in several ways. Sending the projectile in the negative x-direction must be equivalent to reversing the initial trajectory and shifting the origin: so the second half of one trajectory must be the same as the first half of the other. Alternatively, we know that the horizontal and vertical motions are independent. We can eliminate the horizontal motion by viewing the trajectory from a moving frame and we already know that the vertical motion alone is symmetrical about the highest point.

We sketch the expected motion:

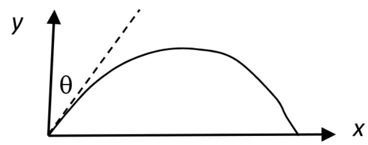

Figure 3.7: Range of a projectile on a flat surface

Now we know what to expect, we can proceed to analyses the motion mathematically. Considering the horizontal motion at constant speed $v\sin\theta$ in time t:

$$x = v\sin\theta t. \tag{3.12}$$

For the vertical motion, we have

$$y = (v\cos\theta)t - \frac{1}{2}gt^2. \tag{3.13}$$

Considering only the vertical motion, we see that $y = 0$ at $t = 0$ and at $t = t_{\text{range}} = \left(\frac{2v\cos\theta}{g}\right)^{\frac{1}{2}}$. The former is of course the time of launch and the latter is the time at which the projectile returns to its launch height. We can find the maximum height by completing the square in (3.13):

$$y = -\frac{1}{2}g\left(t - \frac{v}{g}\cos\theta\right)^2 + \frac{v^2}{2g}\cos^2\theta. \tag{3.14}$$

So y is a maximum when the term in the bracket on the right of (3.14) vanishes, hence at $t = t_{\max} = \frac{v}{g}\cos\theta$. At this time, the height of the projectile, from (3.14), is $v^2 \cos^2\theta / 2g$.

Alternatively, we can use calculus. Differentiating y in (3.13) with respect to t, we get

$$\frac{dy}{dt} = v\cos\theta - gt. \tag{3.15}$$

from which we obtain dy/dt = 0 at

$$t = t_{\max} = \frac{v}{g}\cos\theta.$$

Note that physically dy/dt = 0 means that the vertical speed is zero at the maximum height, which is correct. It also suggests yet another approach: we can use the constant acceleration formula for speed, $v = u + at$, to obtain (3.15) directly. This gives us $t_{\max}$ and hence $t_{\text{range}} = 2\,t_{\max}$ by symmetry.

To calculate the range, we put $t = t_{\text{range}}$ in (3.12) to get

$$x_{\text{range}} = v\sin\theta t_{\text{range}} = \frac{2v^2}{g}\cos\theta\sin\theta = \frac{v^2}{g}\sin 2\theta.$$

At this point, we should check the dimensions of the result: $[v^2]/[g]$ has dimensions $L^2\ T^{-2}/L\ T^{-2} = L$, which is correct. We could guess that the range is of the form v^2/g, because this is here the only quantity with the dimensions of length that enters the problem; but the dimensionless factor, $\sin 2\theta$, could be obtained only by calculation.

This result enables us to calculate the angle of projection to obtain the maximum range: the maximum value of $\sin 2\theta$ is 1 when $\theta = \pi/4$ or 45°. Thus, v^2/g is the maximum range.

Finally, we look at the geometrical form of the trajectory. Equations (3.12) and (3.13) are parametric equations for the trajectory. We can obtain the direct form by eliminating the parameter t:

$$y = \cot\theta\, x - \frac{1}{2} g / v^2 \left(\frac{x}{\sin\theta}\right)^2. \tag{3.16}$$

This is the equation of a parabola. It is perhaps clearer if we complete the square:

$$y = -\frac{g}{2}\left(\frac{x}{v\sin\theta} - \frac{v}{g}\cos\theta\right)^2 + \frac{v^2}{2g}(\cos\theta)^2,$$

which once more gives us the maximum height, $y_{max} = (v^2/2g)\cos^2\theta$ at $x_{max} = (v^2/g)\sin\theta\cos\theta$, from which we can deduce the range $x_{range} = 2\,x_{max}$.

Example 3: The range of a projectile on a slope

We consider next the length of a ski jump assuming a given angle of launch θ and a given constant slope of the jump α (Figure 3.8).

What do we expect? Initially, the projectile cannot "know" the slope of the surface: it will move exactly as if it were launched on a horizontal plane. Only when the projectile reaches $y = 0$ does it become apparent that the surface is not there. So the motion is a continuation of the parabola until it hits the slope. This turns the problem into one of the intersection of two curves, the trajectory and the slope. We begin by sketching this in Figure 3.8.

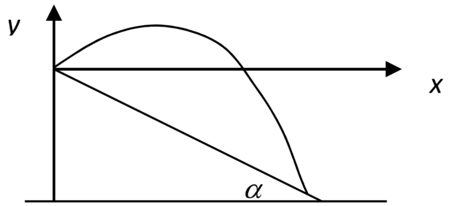

Figure 3.8: Range on a slope

Next we find the curves. We already know the equation of the trajectory (Equation (3.16))

$$y = \cot\theta\, x - \frac{1}{2} g / v^2 \left(\frac{x}{\sin\theta}\right)^2.$$

The equation of the slope is

$$y = -x \tan\alpha.$$

The two intersect when

$$-x\tan\alpha = \cot\theta\, x - \frac{1}{2} g / v^2 \left(\frac{x}{\sin\theta}\right)^2.$$

Thus, the intersection point is either $x = 0$ (which is obviously correct) or

$$x = x_{\text{range}} = \frac{2v^2}{g}(\tan\alpha + \cot\theta)\sin^2\theta.$$

As a check, if $\alpha = 0$, we regain our previous result for x_{range}. Note that this is not the length of the jump, which is measured down the slope: the range down the slope is $R = x_{\text{range}} / \tan\alpha$.

But now the launch angle for maximum range is no longer 45°. Again, before we do the calculation, what do we expect? In the extreme case that the slope is almost vertical, we can see that the range will be larger if the launch angle is shallower giving a higher horizontal component.

To find the actual angle, we have to differentiate and set the result to zero:

$$\frac{dx_{range}}{d\theta} = \frac{v^2}{g}(2(\tan\alpha + \cot\theta)\sin\theta\cos\theta - \sin^2\theta \operatorname{cosec}^2\theta) = 0.$$

Multiplying out terms in the bracket and using the double angle formulae gives

$$\tan\alpha \sin 2\theta + 2\cos^2\theta - 1 = \tan\alpha \sin 2\theta + \cos 2\theta = 0,$$

from which $\tan 2\theta = -\cot\alpha$ or $\tan 2\theta = \tan\left(\alpha + \frac{\pi}{2}\right)$. Therefore, the maximum range occurs for

$$\theta = \frac{\alpha}{2} + \frac{\pi}{4}.$$

You might guess this from the case where $\alpha = \pi/2$, for which the launch should be horizontal (so we have to add $\alpha/2 = \pi/4$ to $\pi/4$ to get $\theta = \pi/2$), but we need the calculation to verify this.

Solution of Problem 2:

We want to find the angle of launch for the maximum range from a tower. This is a much harder problem than those we have tackled so far. It is not the same as the range obtained at 45° from the top of the tower nor is it obtained by launching at a 45° angle from a point behind the tower. A slightly shallower launch from further behind the tower that passes through the top of the tower might land at a larger distance from the tower, even though it is not a maximum range from the point of launch. We shall go through the calculation to the point where we can show that the angle for the maximum range is somewhat <45° to the vertical.

We will use the same axes as in Example 2 with the origin of coordinates at the top of the tower and y measured vertically upward. Then we can write down the horizontal and vertical motion as before:

$$x = vt\sin\theta; \quad y = vt\cos\theta - \frac{1}{2}gt^2.$$

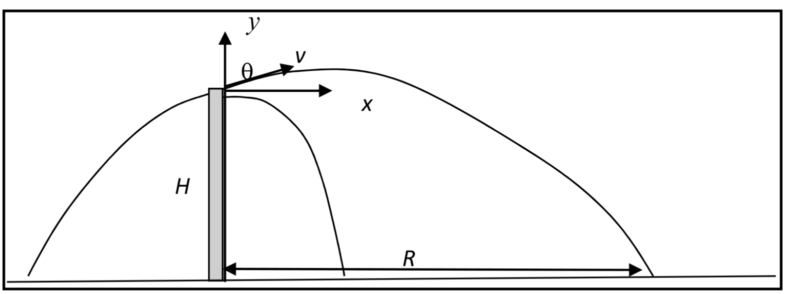

Figure 3.9: Range from a tower

The range is given by the value of $x = R$ when $y = -H$. Let the time when this occurs be $t = T$. Then

$$R = vT\sin\theta, \tag{3.17}$$

and

$$-H = vT\cos\theta - \frac{1}{2}gT^2. \tag{3.18}$$

We have taken some care to distinguish the current coordinates (x, y, t) from the values associated with the range (R, $-H$, T) to be clear about what is constant and what is variable. The height H is fixed, so (3.18) gives us the time T as a function of the launch angle θ. Equation (3.17) then gives the range as a function of θ, which is the quantity we are seeking to maximize. We could try to eliminate T and differentiate R to find the maximum. Often, it is easier to differentiate first and eliminate $dT/d\theta$ and T because the equations will be linear in $dT/d\theta$. Thus, from (3.17)

$$0 = \frac{dR}{d\theta} = vT\cos\theta + v\sin\theta\frac{dT}{d\theta},$$

so

$$\frac{dT}{d\theta} = -T\cot\theta.$$

Also from (3.18):

$$0 = -\frac{dH}{d\theta} = -vT\sin\theta + v\cos\theta\frac{dT}{d\theta} - gT\frac{dT}{d\theta}.$$

Eliminating $\frac{dT}{d\theta}$, we obtain a simple equation for T as a function of θ:

$$T = \frac{v}{g}\sec\theta. \tag{3.19}$$

Now (3.17) will give the range in terms of θ. But we do not yet know θ. However, using (3.19) in (3.18) will give us θ: substituting for T gives a simple equation for $\sin\theta$ that can be solved to yield:

$$\cos\theta = \frac{1}{\sqrt{2}}\left(\frac{1}{1+\frac{gH}{v^2}}\right)^{\frac{1}{2}} < \frac{1}{\sqrt{2}}. \tag{3.20}$$

Thus, $\theta > 45°$. This method requires a lot less algebra than solving for $T(\theta)$ and setting $dR/d\theta = 0$ explicitly.

We can check that $\theta \to 45°$ as $H \to 0$, which recovers our previous result. Furthermore, we can see that the height of the tower starts to make a significant difference only if $H > v^2/g$ or if $v < \sqrt{gH}$. This suggests a different problem: how does the range vary with height for a given angle – in other words, how much range advantage do you gain by releasing a projectile from a height?

Let us do this in the case that θ is adjusted for the maximum range at each height. We can avoid solving (3.18) again because we already know that for this case, T is given by (3.19); also θ is given by (3.20). Thus, the expression (3.17) for the range with these substitutions simplifies to

$$R = \frac{v^2}{g}\left(1 + \frac{2gH}{v^2}\right)^{\frac{1}{2}}. \tag{3.21}$$

So we find also that the height makes a significant difference to the range only if $H > v^2/g$. If $H >> v^2/g$, we can neglect the 1 in the square root, whence $R = \left(\frac{v^2}{g}H\right)^{\frac{1}{2}}$, the geometric mean of the two length scales v^2/g and H. In this case, $\cos\theta \sim 0$ or $\theta \sim \pi/2$, and we are back to the case of an approximately horizontal launch. In other words, once the tower is high enough to make a big difference to the range, the best angle of launch is not very different from horizontal.

Another way of looking at (3.21) is to write it in terms of the parameter $v^2/gH = \lambda$ (say):

$$\frac{R}{H} = \lambda\left(1 + \frac{2}{\lambda}\right)^{\frac{1}{2}}. \tag{3.22}$$

Thus, R is smaller or larger than H by a factor that depends on λ. We can sketch this dependence: for small λ, R/H increases approximately as $\lambda^{1/2}$ changing to an approximately linear dependence for large values. Figure 3.10 shows this dependence.

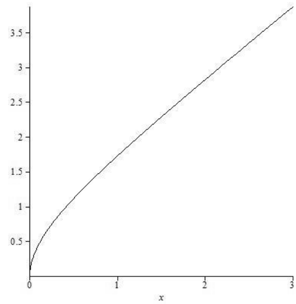

Figure 3.10: A plot of $x\,(1+2/x)^{1/2}$ against x (Equation (3.22)) showing the change from $y \sim x^{1/2}$ to $y \sim x$

3.7 APPROXIMATE SOLUTIONS

Suppose we could not solve problem 2 exactly. We should then try to identify a small parameter in terms of which we can find an approximate solution. This is an important technique, so let us illustrate it in this case (even though we know an exact solution). Return to Equations (3.17) and (3.18), our first task is to identify a small parameter. If something is going to be small, then it has to be dimensionless (all dimensional quantities are "large" if you measure them in "small" units). This suggests that in order to find a small parameter, we should make these equations dimensionless. So we define a new time variable and a new range:

$$\tau = \frac{Tg}{v},$$

$$\rho = \frac{R}{H}.$$

Notice the choice of notation that reminds us of the origin of each of the variables as a time and a length (ideally, we would have used t and r but t is already in use, so we use τ and a matching Greek ρ; it is never a waste of time to think about what to call a variable, since it often saves the effort of having to recall its physical significance). In terms of these dimensionless variables, Equations (3.17) and (3.18) become

$$\rho = \frac{v^2}{gH}\tau \sin\theta, \tag{3.23}$$

$$\frac{v^2}{2gH}\tau^2 - \frac{v^2}{gH}\tau \cos\theta - 1 = 0. \tag{3.24}$$

Thus, the dimensionless parameter is identified as $\lambda = \frac{v^2}{gH}$. Solving (3.24) for τ gives us

$$\tau = \cos\theta + (\cos^2\theta + \frac{2}{\lambda})^{\frac{1}{2}}.$$

There are two following limits:

(i) if $\lambda \to 0$, then $\tau \to \left(\frac{2}{\lambda}\right)^{\frac{1}{2}}$ and then, from (3.23), $\rho \to \sqrt{2\lambda}\sin\theta$. Thus, the maximum range is $\rho = \sqrt{2\lambda}$ at $\theta = \pi/2$ and

(ii) if $\lambda \to \infty$, $\tau \sim \cos\theta + \cos\theta\left[1 + \frac{2}{\lambda\cos^2\theta}\right]^{\frac{1}{2}} \sim 2\cos\theta + 1/(\lambda\cos\theta)$ and $\rho \sim 2\lambda\sin\theta\cos\theta + \tan\theta$. Since $\lambda \to \infty$, we can ignore $\tan\theta$ so the maximum value of ρ is $\rho \sim \lambda$.

This agrees with (3.22) in the two limits $\lambda \to 0$ and $\lambda \to \infty$.

3.8 AIR RESISTANCE

Now suppose we add air resistance to a projectile problem. Will the range be larger because the time of flight is longer? It seems unlikely. Is the time of flight longer in fact, since the maximum height will be less, so the downward journey might take less time rather than more? We seem to need a calculation to solve the problem. However, several issues appear to present themselves. First, with air resistance we cannot separate out the horizontal and the vertical motions, because the component of the resistance in both the horizontal and the vertical directions will depend on the overall speed. Thus, second, it seems that we have to make a model of the forces on the system, and that it is therefore not simply a problem that can be solved in kinematics. On the other hand, whether the range shrinks in the presence of resistance should not really depend on a model for the resistance, but simply on its decelerating effect.

So here is a trick. Suppose we look at the motion from a frame of reference moving with the horizontal motion of the projective without air resistance. In this frame of reference, there is only a vertical motion: the projectile appears to go straight up and come straight down. If we now look at the motion with resistance from this frame, the projectile must always appear to be moving backward. Thus, the range must be shorter than without resistance.

3.9 ADDITION OF ACCELERATIONS

In principle, we can consider the case of constant acceleration in both the horizontal and the vertical directions. An example would be a particle accelerated in a horizontal electric field while falling under gravity. Let us calculate the angle of projection that now gives the maximum range. One might argue that increasing the angle of projection (closer to vertical) will increase the time of flight allowing more time for the horizontal acceleration to act; alternatively, we can afford to decrease the angle to give a greater initial horizontal component because the extra horizontal acceleration will more than take up the slack. So in this case, only the calculation will tell us the correct answer.

For the horizontal motion, we now have

$$x = ut\sin\theta + \frac{1}{2}at^2, \tag{3.25}$$

and for the vertical motion, as usual

$$y = ut\cos\theta - \frac{1}{2}gt^2, \tag{3.26}$$

Putting $y = 0$, we get

$$t_{\text{range}} = 2\frac{u}{g}\cos\theta,$$

and hence,

$$R = 2\frac{u^2}{g}\cos\theta\,\sin\theta + 2\frac{u^2 a}{g^2}\cos^2\theta.$$

To get the maximum, we put $\frac{dR}{d\theta} = 0$, to get

$$0 = \frac{2u^2}{g}\cos 2\theta - \frac{2u^2 a}{g^2}\sin 2\theta,$$

or

$$\theta = \frac{1}{2}\tan^{-1}\frac{g}{a}.$$

First we check that this agrees with the result we already have for constant speed in the x-direction: if $a \to 0$, we recover $\theta = \pi/4$. And we find that for an $a \neq 0$, the acceleration in the x-direction leads to a reduction in the angle of launch $(\tan^{-1}\frac{g}{a} < \pi / 2)$.

We can solve for the shape of the trajectory in space by eliminating t from (3.25) and (3.26). If $t > \frac{2u}{a}\sin\theta$ and $t > \frac{2u}{g}\cos\theta$, we can approximate the trajectory as $y = -x$.

3.10 OTHER FORMS OF ACCELERATION

We shall see in Section 5.5 that a body supplied with a constant power has an acceleration $a = \frac{k}{v'}$ where k is a constant. We can solve the kinematics of this situation by putting $a = v\,\frac{dv}{ds}$. From

$$v\frac{dv}{ds} = \frac{k}{v'}$$

separating the variables and integrating gives

$$\frac{1}{3}v^3 = ks, \tag{3.27}$$

assuming the body starts from rest at $s = 0$. Putting $v = \dfrac{ds}{dt}$ and integrating again, we find

$$s \propto t^{\frac{3}{2}}$$

again assuming that $s = 0$ at $t = 0$. This would be an appropriate approximate model for drag racers.

3.11 CHAPTER SUMMARY

- Kinematics deals with the description of the motion of bodies without regard to the forces required to sustain such motion.
- For motion in one dimension
  i) at constant speed: $s = vt$ is the area under a graph of velocity–time graph, and $v = s/t$ is the slope of a distance–time graph.
  ii) at constant acceleration, $a = v/t$, $s = ut + ½\,at^2$ is the area under a velocity–time graph, from which $v^2 = u^2 + 2as$.
- For motion in two dimensions, horizontal and vertical motions are independent (in the absence of air resistance); thus velocities add as vectors and accelerations add as vectors.
- General formulae for distance and speed can be deduced from $a = \mathrm{d}^2x/\mathrm{d}t^2$ and $a = v\mathrm{d}v/\mathrm{d}x$.
- Models should be expressed in dimensionless variables to identify small parameters.

3.12 EXERCISES

1. A sprint race can be considered to consist of a phase of constant acceleration a lasting for T seconds followed by a run at constant speed v to the finish line. Write down the relationship for distance x in terms of a, T and total time t, and use the data below to find a, v, and T.

   Record for 60 m sprint = 6.39 s
   Record for 100 m sprint = 9.58 s

2. Usain Bolt of Jamaica won the Olympic gold medal in the men's 100 m in a world record of 9.69 s. He then broke that record to win the World Championship final in Berlin in 9.58 s. In the Olympics, he slowed up in the final 5 m. Assuming that otherwise the two records would have been the same and that he ran at a constant speed except for that final 5 m, what was his deceleration (assumed constant)? (the more accurate model of question 1 would give a better estimate)

3. A greyhound track has two straight sides joined by semi-circular arcs. A (mechanical) hare moves at constant speed u round the track. The dogs are released when the hare is ahead of them by a distance d. If the dogs were to run with constant acceleration, a, how long would it take them to catch the hare? Guessing some reasonable values, estimate this time.

4. The acceleration due to gravity on the Moon is 1/6th of that on the Earth. By what factor does this change the maximum range of a projectile launched with a given speed?

5. A package is dropped from a plane moving with speed v at height h. Neglecting air resistance, how far from the intended landing spot should the package be dropped?

6. The musician Percy Grainger is said to have claimed that he could hit a tennis ball over a house from the front garden and run round to catch it in the back garden. Is this feasible?

CHAPTER 4

Energy

For our problem for this chapter, we return to the pyramid of Giza:

Problem:

We can assume that the Great Pyramid of Giza was built during the reign of the pharaoh Kufu which lasted for 23 years. It has a square base of side 230.4 m and an original height of 146.7 m. Can we estimate how many laborers were required?

To address this problem, we need to introduce the concepts of *work* and *potential energy*.

4.1 WORK

It is important to realize that in physics today, we have no knowledge of what energy is.

Richard Feynman was a Nobel prize winning physicist with a great interest in physics education. The quotation above is taken from his famous set of lectures. The fact that "we have no knowledge of what energy is" is unfortunate, because it is this property of physical agents from which their interactions follow. I would go so far as to say that the fundamental laws of physics are specified in terms of the various forms of energy and the attempts to unify physics can be described as attempts to specify a single form of energy as the theory of everything.[1] This is the reason that we choose to start from a hypothesis for the mathematical formulation of energy, rather than the more usual route through Newton's laws of motion. In fact, we shall derive Newton's laws in the next chapter.

In fact, most science textbooks define energy as: "energy is the capacity to do work," which is indeed what Feynman goes on to do. This does not tell us what energy is, but it does tell us how we can use the concept. Strictly speaking, the capacity to do work defines *free* energy, but if we confine ourselves to mechanical systems, where agents do not have internal random motions, the concepts of energy and free energy amount to the same thing.

If we are going to make use of this definition of energy, we need a definition of *work*. If a constant force F moves a body through a distance s in a straight line, we define the work done as

$$\text{work} = \text{force} \times \text{distance},$$

or

$$\Omega = Fs. \tag{4.1}$$

If F is measured in Newtons and s in meters, then Ω is given in Joules (we use Ω as the symbol for work so as not to cause confusion with W for weight).

4.2 KINETIC ENERGY AND WORK

Next we can relate the work done on a body to the gain in its energy of motion or its *kinetic energy*.

Consider the free fall of a weight under gravity from rest through a height h. From the formulae for constant acceleration, we have

$$v^2 = 2gh. \tag{4.2}$$

[1] The *Standard Model* of particle physics, which encompasses all known physics (except gravity), describes the world in terms of an expression for the sum of the energies of all the fundamental particles together with their energies of interaction.

To relate this to work, we need the force on the body. On a static body, we know the force of gravity is the weight W of the body. Thus, as we release the body from rest, the force on the body is W.

We also know that we can describe the motion from any frame of reference moving with constant velocity. Therefore, there is nothing special about the initial position: the body moving at speed v seen from the initial frame will appear momentarily at rest in a frame falling with speed v. Therefore, the force on the body is its weight W at all times during its descent.

Thus, the work done on the body in falling a distance h is $\Omega = Wh$. According to our definition, the energy acquired by the body must be related to Ω. Since we cannot have a theory without assuming something, let us assume that the two are directly proportional:

$$\Omega \propto v^2.$$

The validity of the theory this leads to will have to be tested against experiment. With hindsight, we choose not to investigate the consequences of assuming Ω to be a more general function of v^2. It may seem rather odd to define mass in this way; as we said above, more conventionally, it is defined implicitly through Newton's laws of motion and instrumentally via the momentum balance. Our approach here reflects the contemporary approach to fundamental theories that start from an expression for the relevant type of energy.

For the constant of proportionality, we choose $m/2$, so

$$\Omega = Wh = \frac{1}{2}mv^2. \tag{4.3}$$

Thus, the energy of motion, or kinetic energy, is

$$E_K = \frac{1}{2}mv^2. \tag{4.4}$$

In fact, there are some fundamental constraints on the choice of the expression for E_K in Equation (4.4). The expression must be independent of the origin of time, so it must be unchanged if we make the transformation of variables $t \to t' + c$, where c is a constant. Thus, we do not expect to see an explicit reference to time t in a fundamental expression for energy. So the constant of proportionality cannot explicitly depend on time. Furthermore, we expect the fundamental laws to be reversible, hence not to specify an arrow of time, since our observation of the world suggests that irreversibility is associated with dissipation, which is not a property of individual particles. Thus, E_K must remain unchanged if we let $t \rightarrow -t$, and therefore, it can contain only even powers of v.

4.3 DEFINITION OF MASS

From (4.3) and (4.2), we have

$$Wh = mgh;$$

thus, the parameter m is defined as

$$m = \frac{W}{g}.$$

We call this the mass of the body.

Notice that we have not defined what mass *is*: the parameter m will have to be identified with a measurable quantity by comparison with experiment. Clearly, we want the energy to be an extensive quantity (so doubling the size of the body doubles the energy, other things being equal). So m must account for the amount of material in the body.

The definition of mass was troublesome to Newton, who defined it as the quantity of matter, a definition that one might consider to be not entirely transparent. In his influential exegesis of Newtonian mechanics, Ernst Mach defined mass as a measureable quantity in the context of a ballistic balance, essentially by measuring the quantity mv.[2] Here we have defined mass as the parameter $m = W/g$: the mass of a body is determined by weighing it and dividing by the local acceleration due to gravity, which is the "everyday" definition. Since mass is a property of a body (as it turns out) and not of the environment, the

[2] Ernst Mach, *The Science of Mechanics*.

value of m for a body is fixed once for all independently of the actual local value of the gravitational field (including zero). Of course, this is not an operational definition of mass in the absence of gravity nor does it strictly place mass as a logically primitive concept. For this, we should return to Mach and the momentum balance.

We cannot prove any of these definitions: we simply have to determine if they are consistent with experiment and observation. We do indeed find consistency provided that we are dealing with bodies moving at speeds much less than that of light. That suggests that it is unlikely that alternatives to Newtonian mechanics would be successful, although such attempts exist. For higher speeds, relativity theory provides a different starting point.

With our definition of mass, we can now introduce the standard SI units. Alongside the meter for distance and the second for time, we have the kilogram for mass:

if g *is the local acceleration of gravity, a kilogram is the mass of a body that weighs* g *N.*

We should emphasize again that this is not how SI units are actually defined. Rather the kilogram used to be taken as a fundamental quantity, relating to a platinum object in Paris, and force (the Newton) taken as a derived unit. As of May 2019, the kilogram is defined by taking the fixed numerical value of the Planck constant, h, to be $6.62607015 \times 10^{-34}$ when expressed in the unit J s, which is equal to kg m^2 s^{-1}, where the meter and the second are defined in terms of the speed of light, c, and the hyperfine transition frequency of the caesium-133 atom, $\Delta\nu$, respectively. Nevertheless, in practice, we compare masses by weighing, just as we are doing here.

4.4 WORK AND POTENTIAL ENERGY

For a constant force F acting on an agent over a distance s, we defined a quantity Fs, as the magnitude of the work done on the agent by the external force. If the force opposes the motion we want this to represent a gain in energy of the body. For example, if we raise a body in a gravitational field, in which case, the weight acts in the opposite direction to the motion, we want to say that the body has acquired energy, so we define the work done on the body as $-Fs$. If the force acts in the x-direction and its magnitude is a function of distance, this becomes

$$\Omega = -\int_0^s F(x)\,dx.$$

In general, the force can vary in both magnitude and direction. Suppose that it acts along a curve $\boldsymbol{x} = \boldsymbol{x}(t)$; then the work done is the line integral

$$\Omega = -\int_0^t \boldsymbol{F}(\boldsymbol{x}(t)).\frac{d\boldsymbol{x}}{dt}dt = \int_0^t \boldsymbol{F}\boldsymbol{v}\,dt. \tag{4.5}$$

Consider now pulling a block very slowly up a slope at angle α, so the speed of the block remains infinitesimally small. Then, the forces on the block must almost balance, so $F = W\sin\alpha$. Moving the block up, the slope of a distance s requires work $W\sin\alpha \times s = Wh$, where h is the vertical distance raised.

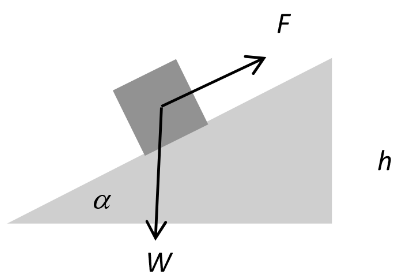

Figure 4.1: Block on an inclined plane

This leads to the interesting observation that the work done is the same whether the block has been pulled up a frictionless slope or raised vertically. The observation can be generalized:

A force is called *conservative* if the integral in (4.5) is independent of the path.

If $\boldsymbol{F}$ is a conservative force, then for a given initial point, the integral in (4.5) defines a function $\Omega = E_P(x)$ of spatial position, $\boldsymbol{x}(s)$. We call such a function the *potential energy* of the system (with respect to the fixed reference point). Usually, the reference point is taken as spatial infinity or as the origin of coordinates

To summarize, the work done, Ω, is always given by (4.5) but is not in general a function of position because it depends on the path; in the case of a conservative force, Equation (4.5) defines a function of position equal to the work done along any path; this is the potential energy that we write as $E_P(x)$.

4.5 CONSERVATIVE FORCES

Equation (4.5) gives us a relation between the force and the work done when the force acts along a path. But it is not possible to use it directly to test if a force is conservative: we cannot evaluate it along every possible path. So we should like to find an instrumental criterion (i.e., one we can apply in practice).

If the force is conservative, so the integral is independent of the path, we can invert the relation (4.5) to get the force in terms of the potential energy. In the force acts in a constant direction, we have

$$E_P(x) = -\int_0^x F(x')\,dx',$$

from which

$$F = -\frac{dE_P}{dx}.$$

The force is the gradient of the potential energy.

In three dimensions,

$$E_P(\boldsymbol{x}) = -\int_0^x \boldsymbol{F}(\boldsymbol{x}')\,d\boldsymbol{x}'$$

implies that

$$\boldsymbol{F} = -\left(\frac{\partial E_P}{\partial x}, \frac{\partial E_P}{\partial y}, \frac{\partial E_P}{\partial z}\right) \equiv -\nabla E_P, \tag{4.6}$$

where the vector ∇E_P (read as "grad E_P") is defined by (4.6), that is, $\nabla f(x) = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$.

Now, if $\boldsymbol{F} = (F_x, F_y, F_z)$ is the gradient of a scalar as in (4.6), and therefore, if $\boldsymbol{F}$ is conservative, then we can show that

$$\nabla \wedge \boldsymbol{F} = \nabla \wedge \nabla E_P(\boldsymbol{x}) = 0, \tag{4.7}$$

where the vector $\nabla \wedge \boldsymbol{F}$ (read as "curl F") is defined by

$$\nabla \wedge \boldsymbol{F} = \left(\frac{\partial F_y}{\partial z} - \frac{\partial F_z}{\partial y}, \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}, \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} \right). \tag{4.8}$$

Equation (4.7) is therefore a necessary condition for a conservative force which we can test in practice.

We can also show that (4.7) is a sufficient condition, that is, if it is satisfied, then $\boldsymbol{F}$ is indeed conservative. The proof (which uses results from vector calculus) is as follows.

Consider the work done along any two paths P_1 and P_2 between points A and B. Then

$$\int_{P_1} F\,dx - \int_{P_2} F\,dx = \int_{P_1 - P_2} F\,dx = \int_s \nabla \wedge F\,dS = 0,$$

where the central integral is taken round the closed curve $P_1 - P_2$ and the second equality follows from Stoke's theorem. It follows that if $\nabla \wedge F = 0$, then the integrals along any two paths are the same and $\boldsymbol{F}$ is conservative, which is what we set out to prove.

Example 1: Any radial force of the form $\boldsymbol{F} = F(\boldsymbol{r})\hat{\boldsymbol{r}}$ is conservative. We have

$$F = \frac{F(r)}{r}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}),$$

from which

$$(\nabla \wedge F)_x = \frac{\partial}{\partial y}\left(\frac{F(r)}{r}z\right) - \frac{\partial}{\partial z}\left(\frac{F(r)}{r}y\right)$$
$$= \frac{F'(r)}{r^2}yz - \frac{F(r)}{r^3}yz - \frac{F'(r)}{r^2}zy + \frac{F(r)}{r^3}zy = 0,$$

with corresponding results for the other components. Forces of this form are therefore conservative.

Just in case, it seems that all forces might be conservative, here is one that is not.

Example 2: A force of the form $\boldsymbol{F} = \boldsymbol{\omega} \wedge \boldsymbol{r}$, with ω a constant vector, is not conservative. Calculating $\nabla \wedge \boldsymbol{F}$ from (4.8), we can show that $\nabla \wedge \boldsymbol{F} = 2\boldsymbol{\omega} \neq 0$. The fact that $\boldsymbol{F}$ is not conservative and is clear if we draw a picture (Figure 4.2). The force is azimuthal about the z-axis, so the work done depends on whether the path is clockwise or counterclockwise.

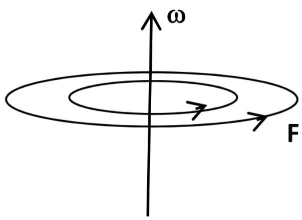

Figure 4.2: Lines of force $\boldsymbol{F} = \boldsymbol{\omega} \wedge \boldsymbol{r}$

4.6 NONCONSERVATIVE FORCES

All of the fundamental forces (gravity, electromagnetism, and the weak and strong nuclear forces) are conservative. Nonconservative forces arise when we consider only a subsystem and not the universe as a whole. For example, consider a body falling through the air. If we were to take account of the interaction of the body with each atom of air, then the total mechanical energy would be conserved: the energy of motion of the falling body would be reduced but that of the air atoms would be raised. Such a calculation would be both impractical and of no interest: our only concern is the falling body. So we replace the effect of the air atoms by a frictional force. We say that the mechanical energy of the falling body is converted irreversibly into heat. The irreversibility is statistical in nature: we never find a macroscopic falling body with increased kinetic energy and cooler air, because this is overwhelmingly unlikely.

4.7 FRICTION AND "ZERO WORK FORCES"

Friction is another obvious example of a nonconservative force. The work done by friction along a path is dissipated as heat and cannot be recovered by returning to the starting point. That much is straightforward. A great deal of confusion arises however when we look in more detail at the role of frictional forces in motion. Let us return to the horse and cart of Chapter 2.

The force accelerating the horse–cart appears to be the reaction to the frictional force from the road on the horse's hooves. According to our discussion so far, this force must do work on the horse and cart to supply the gain in kinetic energy. Now, as everyone knows, the energy to accelerate a cart does not come from the road but from the horse's metabolism (or muscles). This has given rise to the suggestion in some quarters that since all the work done comes from the horse and none (apparently) from the road, some moving forces do no work! But Newton's laws (as with any physical laws) cannot be suspended. So how do we explain this?

Once again, it goes back to an understanding of the microscopic origins of the force we are dealing with. It is easier to start from a simpler situation: that of pushing against a spring, say when an object rebounds from a wall. The compression of the object (and the wall) on impact stores the kinetic energy as elastic energy and returns most of it on rebound. This is exactly what happens as the cart moves along the road. At each instant, there is a compression of the road by the horse's hoofs and a rebound. Energy does indeed flow from the horse to the road and back to the horse and cart. If we look just at the forces on the horse and cart as a body, the force from the road supplies all the kinetic energy (and frictional losses). This does not tell us how the road acquired the energy. As far as the analysis of the forces on the system is concerned, the road could have been supplied with a compression wave that is then ridden by the horse and cart and drives it forward.

4.8 CONSERVATION OF ENERGY

In Section 2, we introduced the idea of kinetic energy by relating it to work, on the assumption that for the falling object, the two were interconvertible without loss. Here we generalize this to all mechanical systems. As our starting point for mechanics, we make the fundamental assumption that the total energy of a system that is subject to conservative forces only is constant. Thus, we assume that the law of conservation of energy

$$E = E_K + E_P = \text{constant}$$

for a conservative system.

It will turn out that this is a profound statement about the physical world. We shall see in Chapter 10 that it is related to the invariance of physical systems in time: that the repetition of a mechanical experiment at any time in the future will give the same results as it does today.

For our body falling from rest under gravity at height h to the ground at $h = 0$, we have

$$0 + mgh = \frac{1}{2}mv^2 + 0$$

in agreement with (4.2).

4.9 UNITS FOR ENERGY

Before we turn to some examples, we shall introduce some useful units for energy in addition to the SI unit of the Joule.

The electron volt (eV) is defined as the work done in moving an electron charge through a potential of 1 V. We have 1 eV = 1.602×10^{-19} J.

The kilowatt hour (kWh): the watt (W) is defined as 1 J s^{-1}. Thus, a kWh is 3.6×10^6 J.

Tons of TNT: 1 gram TNT = 4,184 J so 1 ton of TNT equivalent is equal to 4.2×10^9 J.

4.10 EXAMPLE

The table shows some data on the Airbus A380 super-jumbo airplane taken from the Airbus Website. We can use this to estimate the kinetic energy of the plane at maximum cruising speed with half-empty fuel tanks. We can also estimate the speed of the A380 on takeoff.

http://www.eads-nv.com/1024/en/eads/eads_websites/index/A.html

Normal cruising speed Mach	0.85
Maximum takeoff weight	560,000 kg
Operating empty weight	276,800 kg
Maximum payload (without fuel)	90,800 kg
Maximum cruising speed	Mach 0.89 (945 km h^{-1})
Max thrust of each of four engines	355 kN
Length of runway at max load	2,750 m

The mass of the plane with half-empty fuel tanks is

Mass empty + Payload + Half fuel = 463,800 kg

(where the mass of a full load of fuel is obtained by subtracting the empty weight and the maximum payload from the maximum takeoff weight). A speed of 945 km h^{-1} corresponds to 263 m s^{-1}. So the kinetic energy is

$$\tfrac{1}{2} Mv^2 = 1.6 \times 10^{10}\,\text{J}.$$

This looks like a large number but can we get some idea of how large it is by comparison with something we might know? We can compare it the explosive energy in TNT. We know that 1 ton of TNT has an energy equivalent of 4.2×10^9 J. So 1.6×10^{10} J is equivalent to 3.8 tons TNT or almost four 1,000 lb bombs!

We turn now to an estimate of the speed of the A380 on takeoff from the additional data shown in the table. Note that the quoted "weight" is in fact the mass. The thrust of the four engines gives us the force, F, on the plane. The length of the runway gives us the distance, s, over which this force acts and hence the work done. Assuming this goes into kinetic energy, that is, that there are no losses of energy, we can equate the work done to the gain in energy of $\frac{1}{2}Mv^2$. From this, we can find v as follows:

$$v = \left(\frac{2Fs}{M}\right)^{\frac{1}{2}}$$

$$= \left(\frac{2 \times 4 \times 355000\text{N} \times 2750\text{m}}{560000\text{kg}}\right)^{\frac{1}{2}}$$

$$= 118\text{ms}^{-1}.$$

or about 420 km h^{-1}. The actual takeoff speed is 250 km h^{-1} so there are quite considerable losses.

4.11 BOUND SYSTEMS

Given that the total energy of a system is constant, there are three possibilities: $E < 0, E > 0, E = 0$. If the potential energy is set to be zero at infinity, then these cases correspond, respectively, to bound, unbound, and marginally stable systems. For example, if $E < 0$ (and since $E_K > 0$), then $E_P = E - E_K < 0$ and so the system cannot get to infinity (where $E_P = 0$). The system is therefore confined (or bound).

4.12 VIRTUAL WORK

We can now understand the principal of the lever in terms of work. Suppose that the lever is in balance and displace it through a small angle $\delta\theta$. Then

$$Wx\delta\theta = wX\delta\theta,$$

that is, the work done on one weight by lifting it $x\delta\theta$ equals the work done by the other weight in falling $X\delta\theta$. Thus, there is nothing to be gained by displacing the lever. This is true only if the lever is in equilibrium; otherwise, there would be a net moment and the balance would tip.

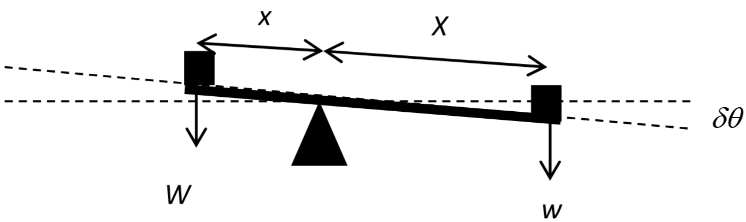

Figure 4.3: Principle of virtual work for a lever

This is a general property of systems in static equilibrium: the net work done on or by a system in equilibrium for a small displacement is zero. This is called the *principle of virtual work*.

Had we not found the equilibrium of the lever by other means, we could have used the principle of virtual work to derive the Archimedes formula for the equality of the moments of the weights. For more complicated structures, we can use the principle to determine the equilibrium configuration.

4.13 ELASTIC ENERGY

The energy stored in a material that has undergone extension or compression is called elastic energy. For a material that is being stretched or compressed in one direction let the original length be L. Suppose that the material is extended by a length x. We define the strain ξ of the material as the quantity

$$\xi = \frac{x}{L}.$$

If the force applied to the material is F and the cross-section area of the material perpendicular to the extension is A, the stress on the material is defined as F/A.

A particularly simple type of elastic material is one that obeys Hooke's law, which states that stress is proportional to strain:

$$\frac{F}{A} = Y\xi,$$

where Y is Young's modulus of the material. We can write this as follows:

$$F = kx,$$

where $k = AY/L$. All materials obey the law for small enough extensions.

Let us calculate how much work is done in stretching the material. We have

$$\Omega = \int_0^x F dx' = \int_0^x \frac{AYx}{L} dx = \frac{AY}{2L} x^2 = \frac{1}{2} kx^2.$$

The energy per unit volume is given by

$$\frac{\Omega}{AL} = \frac{1}{2} Y\xi^2.$$

4.14 EXAMPLE – BUNGEE JUMPING

As an example of elastic energy, let's look at a bungee jump. We shall assume that the bungee cord obeys Hooke's law. The height of the jump must be greater than $L + x$, the original length of the cord plus the extension. The gravitational potential energy lost in a jump of height h by a body of mass M is Mgh, where h must be at least $L + x$ and ignoring the mass of the cord. This must be absorbed by the extension of the cord, so we equate it to $\frac{1}{2}kx^2$. This gives a quadratic equation for x:

$$Mg(L+x)=\frac{1}{2}kx^2. \tag{4.9}$$

A convenient dimensionless parameter here is obviously

$$\lambda = \frac{2Mg}{kL}$$

and our quadratic equation becomes

$$\xi^2 - \lambda\xi - \lambda = 0, \tag{4.10}$$

where $\xi = x/L$. Rather than the exact solution, we can look at two extremes, $\xi << \lambda$ and $\xi \sim \lambda$.

If $\xi >> \lambda$, and we can neglect the second term in the quadratic equation. This gives equation

$$\xi = \sqrt{\lambda},$$

or reinstating physical quantities

$$x = \left(\frac{2MgL}{k}\right)^{\frac{1}{2}}.$$

The extension goes up as the square root of the length of the unstretched cord. For consistency in this case, we must have $\lambda << 1$. This is the case of a stiff cord with a large elastic modulus, and the extension is less than the original length since it corresponds to putting $L + x \sim L$ on the left-hand side of (4.9).

If $\xi \sim \lambda$ and $\lambda > 1$, we can ignore the final term in the quadratic Equation (4.10). Then we get

$$\xi = \lambda$$

or

$$x = \frac{2Mg}{k};$$

the extension is independent of the initial length. Of course, this happens because if the extension is large, we can ignore the small contribution from the original length of cord to the overall height. This approximation corresponds to putting $L \sim 0$ on the left of (4.9).

Of course, the exact solution can be found from the quadratic Equation (4.10):

$$\xi = \frac{1}{2}\left(\lambda + \sqrt{\lambda^2 + 4\lambda}\right),$$

from which we can rederive the two limiting cases for $\lambda >> 1$ and $\lambda << 1$.

4.15 SOLUTION TO THE PROBLEM

The pyramid at Giza is a store of potentially energy of all the work done by the laborers who built it. Conversely, it has the potential to do work if it falls down. Of course, the laborers expended more energy than this, since their ramps (if that is what they used) were not frictionless.

Let the pyramid have semivertex angle α, height h, base side b, and density ρ. Adding the potential energy of each slice, thickness dx a distance x from the top, the potential energy of the pyramid is given by

$$E_P = \int_0^h \rho g(h - x)x^2(2\tan\alpha)^2 dx = \left(\frac{b}{h}\right)^2 \rho g h^4 \int_0^1 (1 - \xi)\xi^2 d\xi$$
$$= \frac{1}{12}\rho g b^2 h^2,$$

or if we put $\xi = x/h$:

$$E_P = \left(\frac{b}{h}\right)^2 \rho g h^4 \int_0^1 (1-\xi)\xi^2 d\xi = \frac{1}{12}\rho g b^2 h^2.$$

In terms of its mass

$$M = \int_0^h \rho x^2 \left(2\tan\alpha\right)^2 dx = \frac{1}{3}\rho h b^2,$$

the potential energy is

$$E_P = \frac{1}{4} Mgh.$$

From the data, and taking the density of stone to be 2,700 kg m^{-3}, the mass is

$$M = 7 \times 10^9 \text{kg}$$

and the potential energy

$$E_P = 2.5 \times 10^{12} \text{J}.$$

How much work can a man do in a day? We know this must be of order of the daily intake of food, say 1,000 kcal ~ 250 kJ (the daily calorie count for laborers might be ~4,000 kcal/day, but the conversion to mechanical work is about 25% efficient). The pyramid could therefore be built in 10^7 man-days. This excludes cutting the stones and building the ramps (if that was how it was done). So the minimum number of men required to build the pyramid in 20 years, or 7,300 days, is 1,400. The report of the Greek historian Herodotus that it took the labor of 100,000 men to build the pyramid seems to be a little bit of fake news.

4.16 CHAPTER SUMMARY

- We can define the mass of a body as its weight divided by the local acceleration due to gravity (recognizing that this is not the fundamental definition)
- The work done by a force $\boldsymbol{F}$ moving through a distance $\boldsymbol{x}$ is defined as the force × distance or $\Omega = \int \boldsymbol{F}\, d\boldsymbol{x}$

- If the force is conservative, the work done is independent of path and defines a potential energy $E_P = \int \boldsymbol{F}\, d\boldsymbol{x}$
- The kinetic energy of a body of mass m moving with speed v is defined as $E_K = \frac{1}{2}mv^2$
- For conservative forces, mechanical energy is conserved: $E = E_K + E_P =$ constant
- The energy of a bound system is negative
- The stress on an elastic body of cross-section A transverse to a force F is F/A.
- The strain of an elastic body of length l extended by an amount x is $\xi = x/l$.
- Hooke's law states that (for small strains) $F/A = Y\xi$, where Y is Young's modulus of the elastic material.
- The energy per unit volume stored in an elastic body is $\frac{1}{2}Y\xi^2$.

4.17 EXERCISES

1. A block of mass M is prevented from sliding down a plane inclined at an angle θ to the horizontal by a horizontal force F. The coefficient of friction between the block and the plane is μ. What is the force required? Under what condition or conditions is $F = 0$? How much work would have to be done to raise the block by a vertical height h allowing for the dissipation by friction.
2. The Golden Gate bridge is supported by steel cables of typical length 100 m exerting a force of 3×10^7 N. What is the minimum radius of a cable? What is the stored energy in a cable?

   The breaking stress of prestressed steel is 1,500 MPa and its Young's modulus is 200 GPa.

3. One model of a leg is a simple pendulum (ignoring the knee joint). Estimate the *maximum kinetic energy of a leg at normal walking speed. What angle of swing would your estimate imply? How high off the ground would the leg swing?

As the leg hits the ground the energy is stored in the Achilles tendon. Estimate the extension of the tendon given the following data.

Young's modulus $Y = 8 \times 10^8$ N m^{-2}
Area A = 89 mm^2
Length l = 250 mm
Maximum extension δl = 15 mm

Assuming this model can also be used for running what is the maximum energy that can be stored in the tendon? Hence, what does the model predict for the maximum running speed? Comment on your estimate.

In this model, how would the length of stride scale with the length of leg?

4. A jumping flea accelerates from rest to a speed of 1 ms^{-1} in 10^{-3} s. The mass of a flea is 0.5×10^{-6} kg and the maximum power from insect muscle is 60 W kg^{-1}. About 20% of the mass of a flea is muscle.

How much kinetic energy does the flea acquire in a jump?
Can muscle alone power the jump?
At the base of each hind leg of the flea is a pad of volume 1.4×10^{-4} mm^3 of resilin, an elastic material with a Young's modulus of 1.7×10^6 Nm^{-2}. Show that enough energy can be stored to power the jump by compressing the resilin pads.

5. Work out (i) the stress and (ii) the stored energy in the Achilles tendon from the following data:

Force exerted on the tendon = 4,700 N
Cross-section = 89 mm^2

Length = 250 mm
Extension = 15 mm

Compare this with the average kinetic energy of a marathon runner.

Show that running shoes make relatively little contribution to the stored energy (rubber has a Young's modulus of <0.36 N mm^{-2}).

CHAPTER 5

MOTION

Problem: The lead shot used in shotgun cartridges consists of small spherical pellets 2–3mm in diameter made by pouring molten lead through a frame suspended in a high tower, a method used since its invention by William Watts in 1782. In order to produce spherical shot, the lead must solidify before the pellet has reached terminal velocity. How high should the tower be?

Figure 5.1: Walters coop shot tower

http://freeaussiestock.com/free/Victoria/Melbourne/slides/walters_coop.htm

5.1 NEWTONIAN DYNAMICS

To attack a problem such as this we have to go beyond kinematics, since we are not given the acceleration, and beyond energy conservation (since energy is dissipated) and look at how the motion of a body depends on the forces acting on it. This general problem was solved by Newton in his three laws of motion. Since these laws form the starting point for dynamics, they cannot be derived and are justified only by agreement with their experimental consequences. However, we shall look first briefly at some of the background to the laws. We shall then see that in simple cases the Newtonian equations of motion can be derived from the conservation of energy. In our final chapter, we shall generalize this approach by making energy the fundamental quantity.

Let us start with a casual observation of everyday life: bodies appear to require a force to keep them in motion. A momentary reflection shows that this is not true: there is no *visible* force acting on a falling body and conversely, on ice, in the absence of an opposing force, you will continue to slide, at least for some time. A better hypothesis might therefore be that bodies have natural states of motion from which they deviate only if subject to some force. This was Aristotle's view: namely, that it required a force to *stop* things from falling. Aristotle also knew about the roughly circular perpetual motion of the planets, so he postulated that the natural motion of "heavenly bodies" was circular, from which they would depart only subject to an applied force. (To explain the different natural motion of terrestrial and celestial bodies, Aristotle postulated that they were made of different materials.)

In order to describe Aristotle's natural motions, we require a standard of rest with respect to which bodies can be said to be moving. Aristotle chose the Earth as his absolute reference frame. If casual observation is all that is available, this is not a stupid theory. To go beyond, it requires some detailed experiments and a willingness to suspend judgment on what one already "knows" (namely that bodies apparently stop moving in the absence of a force).

The first problem with Aristotle's theory arose in the consideration of projectile motion: what is the force that is keeping the projectile moving forward (given that its natural motion is vertical)?

And why does it (apparently) stop moving forward (when it falls)? Various ideas were tried, such as the motive power of air, or the waning of the initial impetus from the thrower. But the real undermining observation was that the Earth is not at rest. If that is the case, why do bodies projected vertically not fall away from the point of projection? As Galileo pointed out, this observation must mean that physics does not distinguish between frames of reference in constant relative motion (and his detractors that it must mean that the Earth does not move, an argument that has been settled in Galileo's favor.) This means that force cannot be the cause of velocity.[1]

So a better hypothesis might be that a force on a body changes its speed. Well, that does not work for the planets, which move with approximately constant speeds but in constantly changing directions. So an even better hypothesis is that a force on a body causes a change in velocity (speed and direction of motion).

This is not yet Newtonian mechanics. We have to postulate the nature of the change in velocity brought about by a force. The simplest proposal is to relate force to acceleration:

$$F = ma,$$

where m is a constant characteristic of the body. This is *Newton's second law*.

This, however, only shifts the previous problem of an absolute standard of rest up one level: to make this law work, we appear to have to specify an absolute state of zero acceleration, with respect to which all nonzero accelerations can be referred. Newton never solved this problem. Instead, he postulated the first law, which simply asserts that there is such a state and we can all find out what it is by validating the second law. So to summarize:

Newton's first law

There exists a state of motion that is unaccelerated that can serve as a reference frame for other states of motion. A body subject

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[1] Suppose that forces caused bodies to move, so that the equation of motion of a body was, say, $F = mv$. Then changing to a frame moving with speed u would change this to $F = mv - mu$: the same force causes a different speed, or $F + mu = mv$, and a fictitious force mu appears, contrary to experience.

to no forces will move with constant velocity (or remain at rest) in this frame of reference.

Once we have identified one such frame, then any observer moving with constant velocity in that frame will provide another such frame of unaccelerated motion. Thus, Newton's first law actually asserts the existence of a class of reference frames (or motions of observers). We call these *inertial frames of reference* or *inertial observers*.

Newton's second law

In an inertial frame of reference, a body of mass m subject to a force $\boldsymbol{F}$ will undergo an acceleration $\boldsymbol{a}$ given by

$$F = ma.$$

The fact that this definition of mass is the same as that in Chapter 3 will become apparent shortly, so we shall not complicate matters by making the distinction.

It should be clear now why the first law is not a consequence of the second: it is trivially true from the second law that if $\boldsymbol{F} = 0$, then $\boldsymbol{a} = 0$, but this is not what the first law asserts. Instead, the first law asserts the existence of a universal frame of reference with zero acceleration. This is a prerequisite for the second law, not a consequence of it.

It may have occurred to the reader that asserting the existence of such a frame (or frames) of reference is not the same as specifying how to find it (or them): who are the inertial observers? We shall take this up again later; for now, it is sufficient to note that the Earth is a good enough inertial frame (because we find that Newton's second law holds if we think of ourselves as at rest) for most engineering purposes, and a frame of reference in which the distant stars are on average nonrotating is a good enough inertial frame for all other purposes (such as the motion of the planets in the solar system).

5.2 EQUATIONS OF MOTION

In standard university courses, we normally think of deriving an energy equation from Newton's equations of motion. This is not how equations of motion are derived in fundamental physics (in, for example, the standard model of particle physics or string theory). We start by postulating a functional form for the energy of the system and derive the laws of motion from that. So that is what we shall do here for a simple one-dimensional system. In more complex situations, with many degrees of freedom, we need special techniques to extract the equations of motion from the energy for each degree of freedom, but the principle is the same and we shall address these situations in Chapter 11.

Thus, following Chapter 4, we start from an energy function $E(x)$ for a particle, with constant mass m, position $x(t)$, moving in one dimension subject to a conservative force $F=-\frac{\mathrm{d}E_{\mathrm{P}}}{\mathrm{d}x}$:

$$E=\frac{1}{2}m\dot{x}^2+E_{\mathrm{P}}(x), \tag{5.1}$$

where $\dot{x}=\frac{\mathrm{d}x}{\mathrm{d}t}$. The first term on the right-hand side is the energy of the free system (without any interactions); the second gives its interactions with the world. In this case, when the world only makes an appearance through parameters in E_{P}, we call E_{P} a potential energy. We now impose the condition that E is conserved:

$$0=\frac{\mathrm{d}E}{\mathrm{d}t}=m\dot{x}\ddot{x}+\frac{\mathrm{d}E_{\mathrm{P}}}{\mathrm{d}x}\dot{x}.$$

Dividing by $\dot{x}$, we get

$$m\ddot{x}=-\frac{\mathrm{d}E_{\mathrm{P}}}{\mathrm{d}x}, \tag{5.2}$$

which is the equation of motion of the particle, or Newton's second law, if we identify $F=-\mathrm{d}E_{\mathrm{P}}/\mathrm{d}x$. Notice that the parameter m appearing in the energy (5.1) also appears in the equation of motion (5.2), justifying our identification of it as the particle mass in both cases.

If the mass of the body is not constant, then we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x})=-\frac{\mathrm{d}E_{\mathrm{P}}}{\mathrm{d}x}.$$

The quantity $p = m\dot{x}$ is the momentum of the particle. Thus, we get the more general for of the second law, namely that

force equals rate of change of momentum.

However, unless we explicitly state otherwise, we shall assume that particle masses are constant.

5.3 AN EXAMPLE

The Chinese F1 grand prix is held on the Shanghai circuit. At corner 14 drivers decelerate from 326 km h^{-1} to 85 km h^{-1}. The circuit map (*http://www.vivaf1.com/shanghai.php*) gives the deceleration as 5.97*g*. What is the braking distance? What is the force exerted given that the minimum mass of an F1 car is 642 kg?

Since we are given speed and a deceleration to find the distance, we use

$$v^2 - u^2 = 2as.$$

Then, converting km h^{-1} to m s^{-1} by dividing by 3.6,

$$s = ((23.6)^2 - (90.5)^2)/(2\times 5.97\times 9.81) = 65.2 \text{ m}.$$

Adding say 60 kg for the mass of the driver, the braking force is given by

$$F = 702\times 5.97 = 4191\,\text{N}.$$

5.4 MOTION IN HIGHER DIMENSIONS

Clearly setting the time derivative of the single quantity E to zero produces only one equation, so we have to modify the approach for motion in two or three dimensions. We shall look at the full theory in Chapter 11, but here is roughly how it works. We have

$$E = \frac{1}{2} m\dot{\boldsymbol{x}}^2 + E_P(\boldsymbol{x})$$

where $\dot{\boldsymbol{x}}^2 = \dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}}$. We now make a small change:

$$0 = \delta E = \frac{1}{2}\left(m\dot{\boldsymbol{x}} \cdot \delta\dot{\boldsymbol{x}} + m\,\delta\dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}}\right) + \nabla E_P \cdot \delta\boldsymbol{x},$$

Where

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

So

$$0 = m\dot{\boldsymbol{x}} \cdot \ddot{\boldsymbol{x}}\delta t + \nabla E_p \cdot \delta\boldsymbol{x}$$

$$= (m\ddot{\boldsymbol{x}} + \nabla E_P) \cdot \delta\boldsymbol{x}$$

Since the increments in (x, y, z) are independent we have, finally

$$m\ddot{\boldsymbol{x}} = -\nabla E_P.$$

5.5 RATE OF DOING WORK

A body, or agent, of mass m and speed v has a stock of energy of motion, or kinetic energy, of $\frac{1}{2}mv^2$. The flow of energy into or out of the body is the rate at which the stock is changing. This turns out to be the rate of doing work on the body by any external force acting on it.

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{x}^2\right) = m\dot{x}\ddot{x} = \dot{x}F.$$

We identify $F\dot{x}$ as the *rate of doing work*, since $F\delta x = F\frac{\delta x}{\delta t}\delta t = F\dot{x}\delta t$ is the work done in time δt. The change in kinetic energy in a displacement δx is the work done, or, equivalently, the rate of change of kinetic energy equals the rate of doing work. The rate of doing work is also called the *power*.

Example: In a race, the dragster with the highest terminal speed at the end of the run wins. We would like to know how the terminal speed depends on the power. In this case, we have a relation between speed v and distance s from Equation (2.8) in Chapter 2

$$\frac{1}{3}v^3 = ks,$$

where k is a constant. Power is defined as the rate of doing work; so for an acceleration, $a = k/v$, the power is

$$P = m\frac{k}{v}v = mk$$

that is, the power is constant. Thus, $k = P/m$ and

$$v = \left(\frac{3Ps}{m}\right)^{\frac{1}{3}}.$$

The terminal speed depends on the power to the 1/3. This is why the power to double the top speed of a road car from (say) 100–200 mph must be increased about eight-fold.

5.6 INERTIAL FORCES

If we transform between frames of reference at constant relative velocity, Newton's laws are unchanged, and therefore, physics is the same for the two observers. To see this, let

$$x' = x - vt$$

be the coordinates of a point viewed by an observer moving with speed v in the positive x direction. Then

$$m\frac{d^2x'}{dt^2} = m\frac{d^2x}{dt^2} = F,$$

and the observer cannot tell which frame of reference they are in by carrying out a Newtonian experiment.

Now consider two frames that are accelerating relatively. We have

$$x' = x + \frac{1}{2}at^2,$$

and

$$m\frac{d^2x'}{dt^2} = m\frac{d^2x}{dt^2} + ma = F + ma.$$

Thus, in the x' frame, the body is subject to an additional force ma. This is a common experience: whenever a vehicle accelerates, the occupants feel an additional force. These additional forces, which appear for accelerating observers, are called *inertial forces*. They are also called "fictitious forces," although they are quite real for the occupants of the vehicle.

This discussion means that care is needed in identifying the frame of reference in which a dynamical system is being described. Newton's laws, with no additional inertial forces, hold only in inertial frames of reference.

5.7 SYSTEMS OF PARTICLES

So far, we have dealt with a single body treated as a particle, having a mass but no extension. To apply the theory to real objects, we should consider an assemblage of particles. We shall find that the translational motion of an extended body can be determined by consideration of the motion of the center of mass. This will justify our application of the equations of motion to extended objects, such as the shot pellet, as if they were point particles.

Suppose we have particles of mass m_i positioned at coordinates $\boldsymbol{x}_i$ with external forces $\boldsymbol{F}_i$ and let the internal force of particle j on particle i be $\boldsymbol{f}_{ij}$. Then for each particle, Newton's second law reads

$$F_i + \sum_{j \neq i} f_{ij} = m_i \ddot{x}_i.$$

Summing over all particles, we get

$$\sum_i \boldsymbol{F}_i + \sum_i \sum_{j \neq i} \boldsymbol{f}_{ij} = \sum_i m_i \ddot{\boldsymbol{x}}_i.$$

But, by Newton's third law, the force particle i exerts on particle j is equal and opposite to that of particle j on particle i: $\boldsymbol{f}_{ij} = -\boldsymbol{f}_{ji} (j \neq i)$; so the sum over the internal forces cancels, and we are left with

$$\sum_i \boldsymbol{F}_i = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\sum_i m_i \boldsymbol{x}_i \right).$$

We define the *center of mass*, analogous to the center of gravity, by

$$M\boldsymbol{X} = \sum_i m_i \boldsymbol{x}_i,$$

where $\mathrm{M} = \Sigma_i m_i$ (Section 2.11). Thus, finally,

$$\sum_i \boldsymbol{F}_i = M\ddot{\boldsymbol{X}},$$

and the translational motion of the body is equivalent to the total external forces acting on the total mass placed at the center of mass. We shall discuss the rotational motion in Chapter 9.

5.8 EXAMPLE: MOTION UNDER AIR RESISTANCE

Air resistance or drag on a body is proportional to the square of its speed. The equation governing the change in speed is therefore

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -kv^2,$$

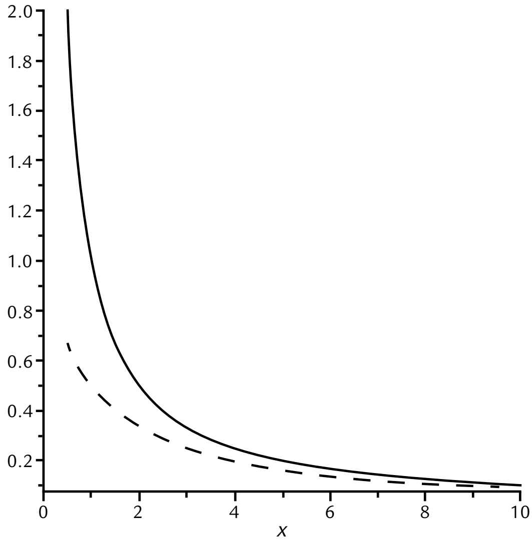


Figure 5.2: Motion under air resistance starting from an initial speed. The dashed line gives the approximate solution at late times

where k is a constant. The solution for a body with an initial speed u is given by

$$\frac{1}{v} - \frac{1}{u} = kt,$$

or

$$v = \frac{u}{ukt + 1}. \tag{5.3}$$

Initially, while $t << 1/uk$, v is approximately constant. If t is very large (much greater than $1/ku$ in fact) we can neglect 1 compared to ukt. So at late times, v is proportional to $1/t$. The graph of v against t must therefore look roughly like the dashed curve in Figure 5.2. The exact solution is given by the full curve.

The speed initially decays at a rate $-ku^2$; if this were to be maintained, the decay would be linear in t and the time for v to reach zero would be

$$\tau = \frac{v_0}{\left(-\frac{\mathrm{d}v}{\mathrm{d}t}\right)_0} = \frac{u}{ku^2} = \frac{1}{ku}.$$

From the exact Solution (5.3), we see that this is the time required for the initial speed to be halved. Thus, assuming a constant decay rate gives us an estimate of the time required for the initial speed to decay significantly. This is an important general result: the timescale of decay of a quantity can be *estimated* by dividing its initial value by its initial rate of decay, as if the rate of decay were constant.

5.9 SKY DIVE

Another example of motion under the combined effect of gravity and air resistance is an attempt to sky dive through the sound barrier in free fall from a height of 39 km above sea level. The difficulty of the problem is the varying density of the atmosphere, which leads to a dependence of the air resistance on height as well as speed. An approximation to the density over the relevant range of height is

$$\rho = \rho_s \left(1 - \frac{z}{a}\right)^{-p},$$

where z is measured down through the atmosphere from the start of the jump in km, ρ_s is the density of air at this point, a = 108 km is a constant and $p \sim 15.9$. The equation of motion is

$$mv\frac{dv}{dz} = mg - K\left(1 - \frac{z}{a}\right)^{-p} v^2,$$

where K is a constant. Putting $\varepsilon = \frac{1}{2}v^2$, the equation of motion becomes

$$\frac{d\varepsilon}{dz} + 2k\left(1\frac{z}{a}\right)^{-p} \varepsilon = g. \tag{5.4}$$

We choose k = K/m such that at sea level (z = 39 km), the terminal speed is 92 km h^{-1}. This gives $k = 0.13\,\text{km}^{-1}$. In these units, $g = 1.27 \times 10^5$ km h^{-2}. There are two regimes: one where gravity dominates over the initial fall, followed by one where air resistance dominates. In the first phase,

$$\varepsilon \approx gz. \tag{5.5}$$

The boundary between the phases, at $z = z_0$, say, is defined by $d\varepsilon/dz = 0$ or

$$2k\left(1\frac{z_0}{a}\right)^{-p}\varepsilon_0 = g$$

or, using (5.5),

$$2k\left(1-\frac{z_0}{a}\right)^{-p} z_0 \approx 1. \tag{5.6}$$

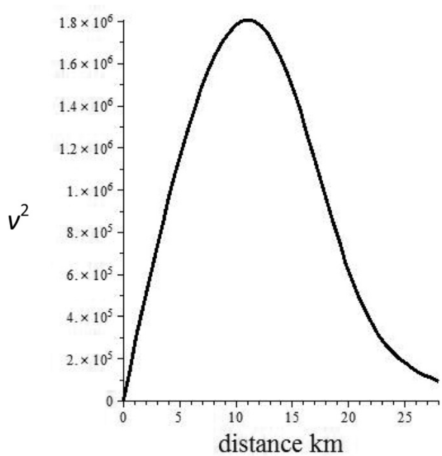


Figure 5.3: Numerical solution for the energy per unit mass plotted against distance fallen for the sky dive (Equation (5.4))

Unfortunately, it is difficult to obtain an approximate solution of this equation (because of the high value of the power p). We can however solve the differential equation numerically and show how this is consistent with (5.6). Figure 5.3 shows that the speed reaches a maximum at $z \sim 10$ km with $\varepsilon \sim 0.9 \times 10^6$ km^2 h^{-2}, corresponding to $v \sim 1.34 \times 10^3$ km h^{-1} or 838 mph. This is roughly consistent with the approximation (5.6) for the turning point of the graph. The official figure for the speed reached was 834 mph.

5.10 TOWER PROBLEM

We are now ready to tackle the problem of the shot tower. We shall present this in terms of several model of increasing detail and accuracy. Throughout, we shall treat the pellet as a single body of mass m.

5.11 MODEL 1

Let us begin by sketching what we expect. In Figure 5.4, we have sketched the velocity of a shot pellet against distance from the top of the tower. To avoid a lot of minus signs, we take distance and speed to be positive in the downward direction. Initially, the speed is small, so the resistance to motion

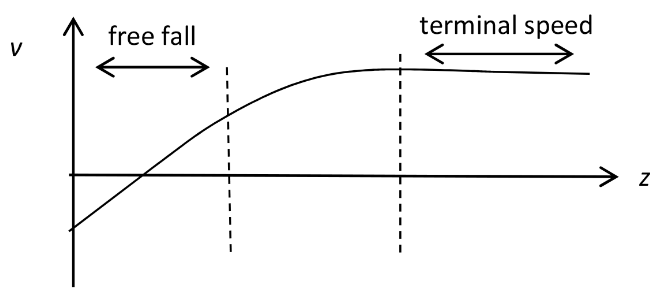


Figure 5.4: The speed (measured downward) plotted against height (measured from the top of the tower) for the falling pellet.

offered by the air is small and the graph must look like constant acceleration. Eventually, air resistance will balance gravity and the pellet will reach a terminal velocity. The graph is quite complicated so to start with we do not attempt to find an exact solution with all factors in play. Instead we approximate the different regimes: an initial phase where the body is in free fall and a final phase where it has reached terminal speed, and an approximation for where the two phases meet.

The constant acceleration phase then is relatively easy. From the graph, we see the relation between speed and height (or distance) so from Chapter 3, we use

$$v\frac{\mathrm{d}v}{\mathrm{d}z} = g,$$

or

$$v^2 = 2gz.$$

5.12 MODEL 2: TERMINAL SPEED

Moving on now to the terminal phase, we need to decide what the force that is acting against gravity. There are two possibilities: viscosity and drag. We can look up the formula for the force in each case or we can estimate it. So as not to interrupt the calculation, let us quote the relevant formulae and see how we could have estimated them later. For the viscous force (resulting from the shear as the air flows past the sides of the drop) on a sphere of radius a moving with speed v through a medium of viscosity η, we have Stokes' formula:

$$F_{\text{visc}} = 6\pi a\eta v. \tag{5.7}$$

For the drag force (resulting from the destruction of momentum of the air impacting a blunt body), we have

$$F_{\text{drag}} = \frac{1}{2}C_D A\rho v^2, \tag{5.8}$$

where ρ is the density of air, $A = \pi a^2$ is the area of the cross-section of the body normal to the flow, and C_D is a constant that depends on the shape of the body but is usually around 0.5.

How do we know which to use? Well, of course, both forces are acting, so we should use the sum of the two. This is rarely a good idea: it complicates matters without offering much illumination. At least to start with, we use whichever is the larger. When v is very small, the viscous term (5.7) must dominate the drag force (5.8) because for v small, $v^2 \ll v$. In fact, we have $F_{\text{drag}} < F_{\text{visc}}$ if

$$v < \frac{12\eta}{C_D a \rho}.$$

Putting in some values for the larger size of shot (a = 3 mm), and taking the density of lead to be 11,000 kg m^{-3}, we get that viscosity dominates drag if

$$v < \frac{12 \times 1.5 \times 10^{-5}}{0.5 \times 0.003 \times 11000} \sim 10^{-2} \,\text{m s}^{-1}. \tag{5.9}$$

So initially (when $v \sim 0$), we have to consider only viscosity. Let us then estimate the viscous term: We have, taking the larger size of shot

$$6\pi a \eta v = 6\pi \times 0.003 \times 1.5 \times 10^{-5} v \sim 10^{-6} v \text{ N}.$$

Thus, there is a gravitational force accelerating the pellet and a viscous force opposing the motion. Let us compare the magnitude of these, we have

$$Mg = 11{,}000 \times \frac{4}{3}\pi a^3 \times 9.8 \sim 10^{-2}\text{N} \gg 6\pi a \eta\, v \text{ if } v < 10^4 \,\text{m s}^{-1}.$$

So in the initial fall, for small speeds, we can clearly neglect viscosity relative to gravity: the initial motion is just free fall under gravity. (One might guess this from the observation that air is not a particularly viscous medium, so drag is almost always more important for macroscopic bodies.) However, from Equation (5.9), the drag will become more important than viscosity once $v > 10^{-2}\,\text{m s}^{-1}$ and will balance the gravitational force once

$$\frac{1}{2} C_D A \rho v^2 = Mg. \tag{5.10}$$

Thus, the maximum speed, or *terminal speed*, v_t, of the shot is given by solving (5.10) for v:

$$v_t = \left(\frac{2Mg}{C_D A \rho}\right)^{\frac{1}{2}} = \left(\frac{8ag\rho_{\text{lead}}}{3C_D \rho_{\text{air}}}\right)^{\frac{1}{2}} \sim \left(\frac{8 \times 0.003 \times 9.81 \times 11{,}000}{3 \times 0.5 \times 1.29}\right)^{\frac{1}{2}} \sim 37 \text{m s}^{-1}.$$

We can *estimate* the distance, s_f, to reach this speed by using the free fall equation:

$$s_f = \frac{v_t^{\,2}}{2g} = \frac{4a\rho_{\text{lead}}}{3C_D\rho_{\text{air}}} \sim 68\,\text{m}.$$

This corresponds to a time

$$t_f = \sqrt{\frac{2s_f}{g}} \sim 4\,\text{s}.$$

Since the body is not freely falling, but accelerating more slowly, this is an underestimate.

To solve the problem posed by the cooling of the shot, we need to consider the time of fall. We can assume that this is set by the time required for the shot to cool, which we suppose for the sake of argument is >1 s (the time in free fall). Thus the fall can be approximated by two phases: one in free fall at constant acceleration, followed by a period at a constant speed of 8 m s^{-1}. The time of fall is thus

$$t_{\text{total}} = t_f + \frac{h}{v_t},$$

where h is the height fallen at the terminal speed or

$$H = h + s_f = \left(t_{\text{total}} - t_f\right)v_t + s_f.$$

5.13 MODEL 3

In fact, this problem can be solved exactly, so we have the opportunity to compare our approximation with the exact solution. The equation of motion is

$$mv\frac{\mathrm{d}v}{\mathrm{d}z} = mg - \frac{1}{2}C_D\rho_a A v^2.$$

Dividing through by the mass of the pellet, we have

$$v\frac{\mathrm{d}v}{\mathrm{d}z} = g - \frac{v^2}{2s_f},$$

where $s_f = \left(\frac{4a}{3C_D}\right)\frac{\rho_L}{\rho_a}$ as above. The equation confirms our earlier analysis that for length scales $<s_f$, gravity is important, and for length scales $>s_f$, the drag term dominates. To see this, we approximate $v\mathrm{d}v/\mathrm{d}z$ as v^2/z. Then

$$v^2 \sim \frac{g}{\frac{1}{z}+\frac{1}{2s_f}}.$$

So for $z < 2s_f$, we can ignore the drag term.

The first thing to do is to tidy up the equation of motion by introducing some dimensionless variables. One reason is that otherwise we would find ourselves writing out the constants such as C_D again and again as we work through the algebra. A more important reason is that the collection of constants obscures the meaning of the equation. We make z dimensionless by dividing by s_f: so define

$$x = \frac{z}{2s_f}.$$

(The factor of 2 is included with hindsight to tidy up the working.) If we now define

$$u = \frac{v}{\sqrt{2s_f g}},$$

the equation of motion takes the dimensionless form

$$\frac{\mathrm{d}u^2}{\mathrm{d}x} = 1 - u^2.$$

The two regimes are now clear. For $u^2 << 1$, we have $u^2 \sim x$ or $u \sim \sqrt{x}$. As $u^2 \to 1$, $\mathrm{d}u^2/\mathrm{d}x \to 0$, so $u^2 = 1$ is the limiting value for u^2.

The equation of motion is a first-order separable differential equation so the exact solution can be obtained by rearrangement and integration:

$$\int \frac{u^2}{1-u^2} \mathrm{d}u^2 = x,$$

or

$$u^2 = 1 - e^{-x}, \tag{5.11}$$

where we have chosen the constants of integration such that $u^2 = 0$ at $x = 0$. Note that $u^2 \to 1$ as $x \to \infty$, and that for finite x, u never actually reaches 1. This means that the terminal speed is never reached exactly. However, for all practical purposes, unless we require a high level of accuracy, we can take terminal speed to be reached at around $x = 1$, as we did in Model 2 above.

We now have to solve for distance as a function of time. Since $\mathrm{d}s/\mathrm{d}t = v$, or, equivalently,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u\sqrt{\frac{2g}{s_f}},$$

we define a timescale $T = \sqrt{(s_f/2g)}$ and put

$$\tau = \frac{t}{T} = t.$$

Then, the equation of motion becomes

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = u = \sqrt{1 - e^{-x}},$$

from (5.11) with the initial conditions $x = 0$ at $\tau = 0$. You might be tempted to think that this cannot be integrated analytically and instead try to solve it numerically using a standard numerical method. The problem would then be that at $x = 0$, we have $\mathrm{d}x/\mathrm{d}t = 0$. Thus, numerically, x never changes and the solution appears to be $x = 0$ for all time. The numerical equation solver needs some help, which we would give it by expanding the solution about $x = 0$ and setting the initial conditions at $t = \delta$. So, for small x, by expanding the exponential, we get

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} \sim x^{\frac{1}{2}},$$

from which

$$\tau = 2x^{\frac{1}{2}}.$$

The initial conditions become $x = (\delta/2)^2$ at $\tau = \delta$ (where δ would then be given some small value, say 10^{-2}).

In fact, the equation can be solved exactly: we get

$$\tau = \int \frac{dx}{\left(1-e^{-x}\right)^{\frac{1}{2}}} = 2\tanh^{-1}\left(1-e^{-x}\right)^{\frac{1}{2}}, \tag{5.12}$$

with the constants of integration chosen to satisfy $\tau = 0$ at $x = 0$.

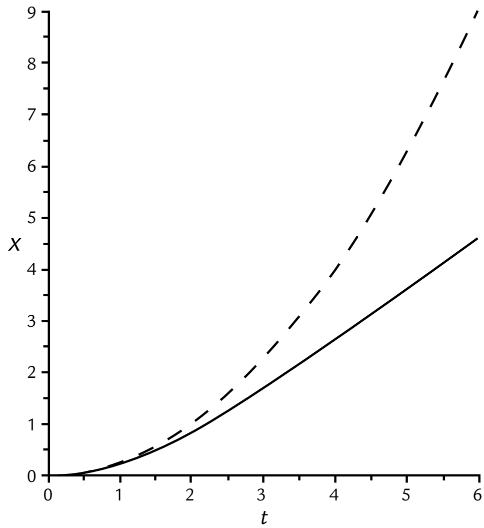


Figure 5.5: Exact solution (solid line) and approximate solution (dashed line) for the time against distance fallen. The exact solution approaches $x = \tau - 2\ln 2$ as $\tau \rightarrow +\infty$

We can invert (5.12) to give x as a function of t:

$$x = -\ln\left(1-\left(\tanh\frac{\tau}{2}\right)^2\right).$$

This is plotted in Figure 5.5. We have also shown on the figure the approximate solution for early times, $x = (\tau/2)^2$. The figure shows how the graph departs from this solution and becomes approximately a straight line $x \sim \tau - 2\ln 2$ around $\tau = 2$. Note how taking drag into account reduces the required height of the tower.

Finally, we should express the solution in physical units:

$$t = \left[\left(\frac{2a}{3gC_D}\right)\frac{\rho_L}{\rho_a}\right]^{\frac{1}{2}} \tanh^{-1}\left(1 - \exp\left(-\frac{3C_D\rho_a z}{8\rho_L a}\right)\right)^{\frac{1}{2}}.$$

You should check that the prefactor has the units of time and that the argument of the exponent is dimensionless.

From this, it is clear that the length scale

$$s'_f = \frac{8\rho_L a}{3C_D\rho_a} = 2s_f$$

divides the behavior of the trajectory into two phases; For $z \lesssim s'_f$, we have approximately free fall and for $z \gtrsim s'_f$ drag starts to become important. This agrees with our previous approximate solution.

Solving for z gives

$$z = -\frac{8a}{3C_D}\frac{\rho_L}{\rho_a}\ln\left\{1 - \left[\tanh\left(\frac{3gC_D}{2a}\frac{\rho_a}{\rho_L}\right)^{\frac{1}{2}} t\right]^2\right\}.$$

Finally, we should put in some numerical values. Shotgun pellets are around 6 mm in diameter ($a = 3$ mm); the density of lead is 11,300 kg m^{-3}; the density of air is 1.29 kg m^{-3}. If the time required for lead to solidify were around 5 s (say), this would give a height of about $4.4s_f$ ~ 250 m, in agreement with the graph of Figure 5.5.

In fact, towers are rather shorter than this because the lead must solidify in the free-fall phase if the pellets are to be spherical, as we see next.

5.14 THE SHAPE OF THE SHOT

The problem asked us to determine the shape of the shot. To do this, we transform our point of view to the reference frame of the shot. In the initial phase, under free fall, in frame of reference of the falling shot the acceleration is zero, so the shot is weightless. What does this mean for the shape? In the absence of gravity, the surface tension of the lead will form the drop into a sphere. In the constant velocity phase, the drag on the drop prevents it from moving (relative to us as observers falling with it, now at constant speed). A drop at rest under gravity, on a table say, adopts a flattened shape. So this is the shape of the shot both in our frame and, of course, in the frame of the tower. Tear shaped raindrops are a myth!

5.15 UPTHRUST

A solid body immersed in a fluid (by which we mean a gas or liquid) displaces an equal volume of the fluid. Prior to the introduction of the solid body, the displaced volume was neither sinking nor rising. The gravitational force on it must therefore have been balanced by the force of the surrounding medium; in other words, any volume of fluid experiences an upthrust equal to its weight. Since this upthrust is the result of the action of the surrounding medium, it must still be present after the solid body is introduced. Thus, we arrive at *Archimedes' Principle*:

A solid body immersed in a fluid experiences an upthrust equal to the weight of the fluid it displaces.

An obvious corollary is that a less dense object will float in a denser fluid.

A body of mass M, density ρ_b, falling in a fluid of density ρ_f experiences an upthrust $\rho_f\left(\frac{M}{\rho_b}\right)g$ and therefore has an effective weight of

$$W_{\text{eff}} = W\left(1-\frac{\rho_f}{\rho_b}\right).$$

Archimedes' Principle generates various paradoxes. Suppose we weigh an object (W) and a tub of water (w) separately. The combined weight is $W + w$. Now place the object in the tub and assume no water is spilt. Is the weight now $W_{eff} + w$? This is impossible because we could generate a perpetual motion machine just by adding and removing the weight (using the upthrust as a driver). The paradox is resolved if we appreciate that the weight cannot be in static equilibrium immersed in the fluid: it will be falling. The fall imparts downward momentum to the fluid which impacts the bottom of the container with a downward force; perhaps this is just sufficient to keep the overall system with a weight W. This seems strange since the body might be falling quite slowly if the medium is more viscous than water.

What happens when the weight hits the bottom then? Does a weight resting on the bottom of a container not experience an upthrust? Indeed, it must be the same argument we used to deduce Archimedes' principle.

For the resolution, we must take into account the rise in the fluid level in the tub. This will increase the pressure at the bottom just enough to compensate for the upthrust on the body.

Another paradox emerges if we try to find the weight of air by accurately weighing a balloon empty and comparing this with the weight of an inflated balloon. We would discover that air is apparently weightless.

In fact, we know that air does have weight. To explain why this attempt to measure the weight of air has failed, we could bring in Archimedes' principle according to which there is an upthrust on the balloon equal to the weight of air it displaces – that is, an upthrust that exactly balances the weight of air inside it.

But you do not need a theory to see this (or more correctly, the theory is only a concise expression of what you already intuitively know). The fallacy is the same as the fact that you cannot see a blue balloon against a blue background. If the filled balloon weighed more than the empty one, then it would fall to the ground. But you cannot make the air inside a region heavier than it was previously by putting a balloon round it.

5.16 SIMPLE HARMONIC MOTION

Newton's laws enable us to find the motion of a body under a given force. One important example is the simple harmonic oscillator. It is useful to have in mind a concrete picture. One example is a body on a spring that is displaced from its equilibrium position (Figure 5.6) assuming that the spring obeys Hooke's law.

In equilibrium, we assume that the spring has an extension x_0 (measured downward), then

$$W = kx_0.$$

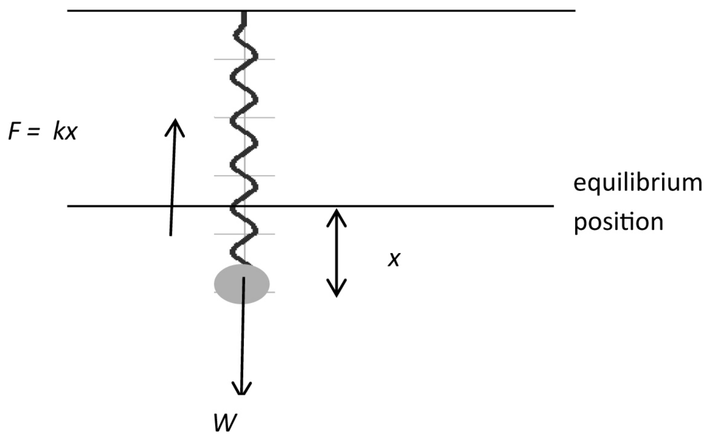


Figure 5.6: Mass on a spring

Once displaced the spring will oscillate; if the total extension of the spring is $x + x_0$, then

$$m\ddot{x} = W - k(x + x_0) = -kx.$$

We can simplify the notation slightly by defining $\omega^2 = k/m$, so the equation of motion becomes

$$\ddot{x} = -\omega^2 x. \tag{5.13}$$

Equation (5.13) defines simple harmonic motion: in general, any system (not just a spring) which obeys this equation of motion is said to be a *simple harmonic oscillator* (SHO) or undergo *simple harmonic motion* (SHM)

As usual we try to describe the motion before turning to the mathematical analysis. The physical picture of a spring tells us that the motion should be oscillatory. The body slows down as it approaches the extremes of the displacement and speeds up as it accelerates through the equilibrium position at $x = 0$. If we start the body from a displacement $x(0)$ at $t = 0$, with some downward speed, and if we measure x as positive in the downward direction, we expect something like Figure 5.7 for the displacement (dashed line) and speed (solid line).

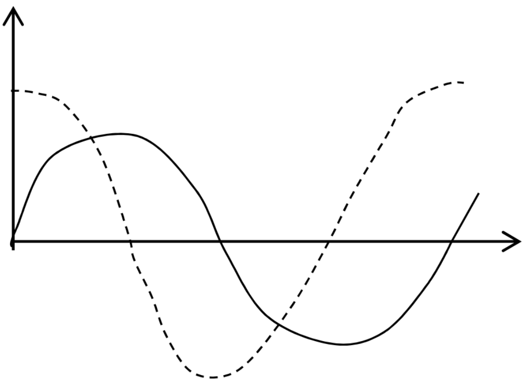


Figure 5.7: Sketch of the expected motion of a mass on a spring starting from a nonzero displacement and zero speed. The speed is a maximum at zero displacement

Turning now to the mathematical analysis, the general solution of Equation (5.13) is

$$x = a\cos\omega t + b\sin\omega t, \tag{5.14}$$

where a and b are arbitrary constants. This can be verified by differentiation:

$$\dot{x} = -a\omega\sin\omega t + b\omega\cos\omega t, \tag{5.15}$$

and hence

$$\ddot{x} = -a\omega^2\cos\omega t - b\omega^2\sin\omega t = -\omega^2(a\cos\omega t + b\sin\omega t) = -\omega^2 x.$$

The constants a and b are fixed by the starting conditions at $t = 0$. For example, we might know that $x = x(0)$ at $t = 0$ and $\dot{x}(0)$. Then at $t = 0$, from (5.14)

$$x(0) = a\cos 0 = a,$$

and from (5.15),

$$0 = b\omega\cos 0 = b\omega$$

so $b = 0$ and $a = x(0)$. Hence, the solution for this set of initial conditions is

$$x = x(0)\cos\omega t.$$

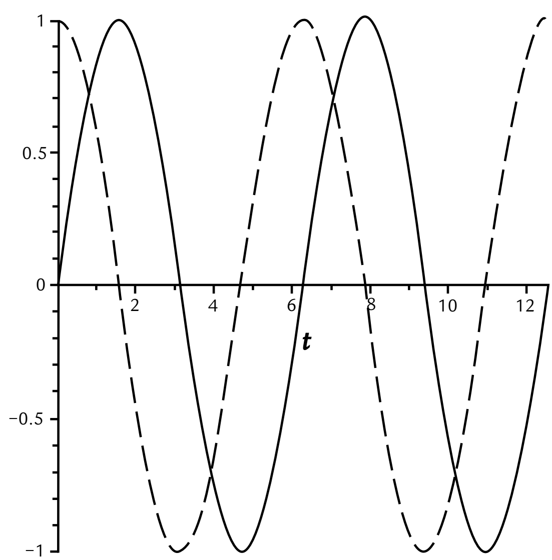


Figure 5.8: The displacement cos*t* (dashed line) and velocity sin*t* (solid line) of a harmonic oscillator started from rest with unit initial displacement

Figure 5.8 illustrates the solution graphically and confirms the general form of our initial expectation. It also shows (from the periodicity properties of the cosine function) that ω is the angular frequency of the oscillation, or, equivalently, that $f = \omega/2\pi$ is the frequency and $T = 1/f = 2\pi/\omega$ is the period.

5.17 WHY SHM IS IMPORTANT

To see why SHM plays such an important role in mechanics consider a body, coordinate x acted upon by a force that varies as some function $F(x)$. Assume that at some point x_0, the force vanishes:

$$F(x_0) = 0.$$

Newton's second law tells us that a body placed at this point (with zero speed) will remain there. Thus, x_0 is an equilibrium point. (There may be more than one.) Now consider a small displacement $x = x_0 + \varepsilon$. Then

$$\ddot{x} = \ddot{\varepsilon} = F(x_0 + \varepsilon) = F(x_0) + \varepsilon \left(\frac{\mathrm{d}F}{\mathrm{d}x} \right)_0 + \ldots,$$

using the Taylor expansion of $F(x_0 + \varepsilon)$. If we neglect higher-order terms, the equation of motion becomes

$$\ddot{\varepsilon} = \left(\frac{\mathrm{d}F}{\mathrm{d}x} \right)_0 \varepsilon.$$

If $\mathrm{d}F/\mathrm{d}x$ at x_0 is negative, we can set it to $-\omega^2$. This will be the case if x_0 is a point of stable equilibrium. For small displacements about a stable equilibrium point, *any* system behaves (approximately) as an SHO. (If $\mathrm{d}F/\mathrm{d}x$ at x_0 is positive, ε will grow exponentially until it can no longer be assumed to be small. In this case, the equilibrium is unstable.)

We shall investigate oscillatory motion more fully in Chapter 8.

5.18 ENERGY OF A HARMONIC OSCILLATOR

Since we view energy as the fundamental dynamical quantity, we would like to derive the equations of motion of a harmonic oscillator from an expression for energy. There are several ways to do this. One general method is to reverse the process that leads us from energy to the equations of motion. For a harmonic oscillator, we have

$$m\ddot{x} = -kx.$$

Multiplying by $\dot{x}$:

$$m\dot{x}\ddot{x} = -kx\dot{x}.$$

We can write this as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{x}^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}kx^2\right),$$

in which form the equation can be integrated:

$$E = \frac{1}{2}m\left(\dot{x}^2 + \frac{1}{2}kx^2\right), \tag{5.16}$$

where E is a constant (which must be the energy since the first contribution to the sum is the known kinetic energy) and $k = m\omega^2$.

Alternatively, we know the kinetic energy is $\frac{1}{2}m\dot{x}^2$ so we have only to find the potential energy from the relation

$$F = -kx = -\frac{\mathrm{d}E_P}{\mathrm{d}x}.$$

Integrating, we get $E_P = \frac{1}{2}kx^2$ and, adding the kinetic energy, a total energy that agrees with (5.16).

5.19 CHAPTER SUMMARY

- Newton's first law: There exists a state of motion that is unaccelerated that can serve as a reference frame for other states of motion. A body subject to no forces will move with constant velocity (or remain at rest) in this frame of reference.
- Newton's second law: In an inertial frame of reference, a body of mass m subject to a force F will undergo an acceleration a given by $F = ma$.

- In a noninertial frame of reference, additional "inertial forces" appear in the equations of motion.
- The equations of motion for a conservative system can be obtained by differentiating the total energy

$$E = \frac{1}{2}m\dot{x}^2 + E_P(x).$$

- The rate of doing work by a force F on a body moving with speed $\boldsymbol{v}$ is $F\,v$.
- The translational motion of the body is equivalent to the total external forces acting on the total mass placed at the center of mass
- Solving Newton's equations of motion usually requires approximations. Relevant approximations can be obtained by introducing dimensionless variables and identifying small parameters.
- Archimedes' Principle: A solid body immersed in a fluid experiences an upthrust equal to the weight of the fluid it displaces.
- The displacement from equilibrium $x(t)$ of an SHO satisfies the equation of motion: $\ddot{x} = -\omega^2 x$.

5.20 EXERCISES

1. The acceleration of a body moving in a straight line with speed v through a certain medium is given as $-kv$. Write down the equation of motion of the body.

Show that such a body, starting with speed u, moves according to

$$x = \frac{u}{k}(1 - e^{-kt})$$

and hence comes to rest in a finite distance u/k.

2. Investigate damped SHM with a damping proportional to rv^4 for small r.

3. In Figure 5.9, the chain on the right-hand slope is more massive than the chain on the left. Therefore, the chain will move clockwise forever. What is the fallacy?

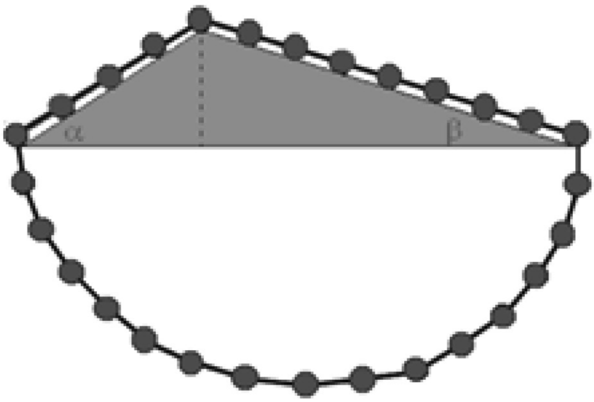


Figure 5.9: Question 3

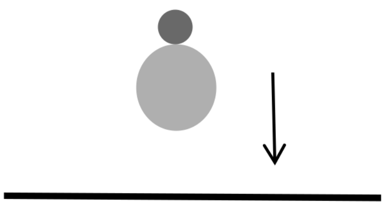
4. Estimate the rate of doing work against drag by a car on a motorway.

5. A helium balloon with a rigid envelope is released from rest in still air. What is its maximum height and how long does it take to reach it? (Party balloons have a radius of 0.1143 m and a mass of 0.0185 kg.)

CHAPTER 6

MOMENTUM

Problem: The figure shows a small ball balanced on a much larger one falling together toward a solid floor. What happens next?



6.1 CONSERVATION

The velocity of a free particle is constant, so its momentum, mv is constant. More generally, from Newton's law in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)=\boldsymbol{F},$$

we see that, if there is no external force, the rate of change of the momentum is zero and hence the momentum is constant whether or not the mass varies.

We can generalize this result to a system of particles. Suppose we have particles of mass m_i positioned at coordinates $\boldsymbol{x}_i$ with external forces $\boldsymbol{F}_i$ and let the internal force of particle j on particle i be $\boldsymbol{f}_{ij}$. Then for each particle, Newton's second law reads

$$\boldsymbol{F}_i + \sum_{j \neq i} \boldsymbol{f}_{ij} = m_i \ddot{\boldsymbol{x}}_i.$$

Summing over all particles, we get

$$\sum_i \boldsymbol{F}_i + \sum_i \sum_{j \neq i} \boldsymbol{f}_{ij} = \sum_i m_i \ddot{\boldsymbol{x}}_i.$$

But $\boldsymbol{f}_{ij} = -\boldsymbol{f}_{ji} (j \neq i)$ by Newton's third law; so the sum over the internal forces cancels and we are left with

$$\sum_i \boldsymbol{F}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i m_i \dot{\boldsymbol{x}}_i \right).$$

If the net external force is zero, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i m_i \dot{\boldsymbol{x}}_i \right) = 0. \tag{6.1}$$

We define the position vector $\mathbf{X}$ of the center of mass (CM) by

$$\sum_i m_i \boldsymbol{x}_i = M\mathbf{X},$$

where M is the total mass. (Compare Equation (2.11); if we arrange for the origin to be at the CM, the net moment of the masses is zero. Note also that the CM is not the same as the center of gravity if the acceleration due to gravity varies across the system, although you are unlikely to need to know this except in examinations.) Thus, (6.1) gives

$$\sum_i m_i \dot{\boldsymbol{x}}_i = \text{constant} = M\dot{\mathbf{X}}.$$

In words, the total momentum is the momentum of the total mass moving with the CM; if a system of particles is subject to no net external force, then the momentum of the system is conserved.

6.2 CONSERVATION AND INVARIANCE

In general, if the energy of the system is independent of position, momentum is conserved. This is trivial to show for a single particle of constant mass subject to conservative forces. We have

$$E = E_K + E_P,$$

with $E_K = \frac{1}{2}mv^2 = \frac{p^2}{2m}$ and $\frac{dE_P}{dx} = 0.$ Thus

$$0 = \frac{dE}{dt} = \frac{p}{m}\frac{dp}{dt} + \frac{dE_P}{dx}\frac{dx}{dt} = \frac{p}{m}\frac{dp}{dt},$$

so

$$\frac{dp}{dt} = 0,$$

and p = constant. We shall show this more generally in Chapter 11.

6.3 IMPULSE

More generally, if now a system is subject to a force over some time interval, we can write

$$\int dp = \int F dt.$$

The left-hand side is the change in momentum over the time interval. We call the right-hand side the impulse. Thus, we have

Change in momentum = Impulse.

We can compare this with the corresponding relation for energy:

Change in energy = Work done.

Example: The speed of a soccer ball can be up to about 30 m s^{-1}. The time of contact between the foot and ball has been measured at 0.05 s, and the mass of a soccer ball is 0.45 kg. What is the force exerted by the player?

We have $F\Delta t = m\Delta v$ in an obvious notation. Thus,

$$F = 0.45 \times \frac{30}{0.05} = 270\,\text{N}.$$

Thus, a player exerts a force of about one-third of their weight.

6.4 COLLISIONS IN ONE DIMENSION

The conservation of momentum is useful in analyzing collisions where there are, by definition, no external forces (only internal forces). In a collision, we are often interested only in the situations before and after the event and not in following the details through the collision.

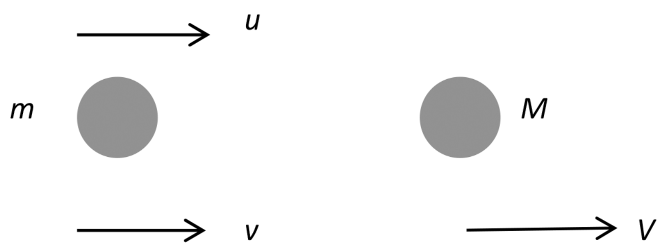


Figure 6.1: A collision with one body (mass *M*) initially at rest

Consider a body of mass m moving with speed u on a frictionless surface in a collision with a body of mass M at rest (Figure 6.1). Let the speeds after collision be v and V, respectively. We assume that the collision conserves energy as well as momentum. Such collisions are termed *elastic*. One case we can solve without calculation is that of equal masses, $M = m$. Since energy is conserved, these collisions are reversible. But in this case, the time reversed collision must look exactly like the original collision (since the masses are identical). Thus, the solution must be $v = 0$ and $V = u$; the moving body comes to rest and the originally stationary body moves off with the speed of the incoming body.

How can we make this symmetry more obvious? If we view the collision in a frame moving with speed $u/2$, we should see incoming particles each with speed $u/2$ (and opposite directions). The symmetry is now clear: the motion of each particle must be reversed in the collision. In the original frame, this corresponds to an interchange of speeds.

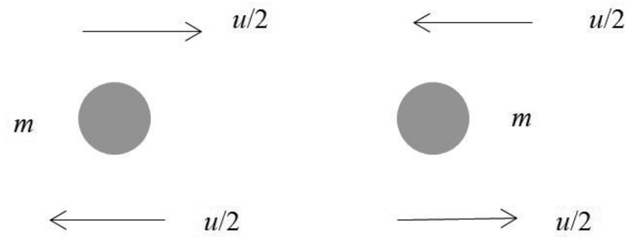


Figure 6.2: The collision of Figure 6.1 with equal mass particles viewed from a frame moving to the right with speed $u/2$

If the masses of the two particles are not equal, we might guess that the CM frame would be a good way to look at the collision: in this frame, the momentum must be zero before the collision and therefore zero after the collision. The only way this can happen is for the particle velocities to be reversed. Figure 6.2 shows the result.

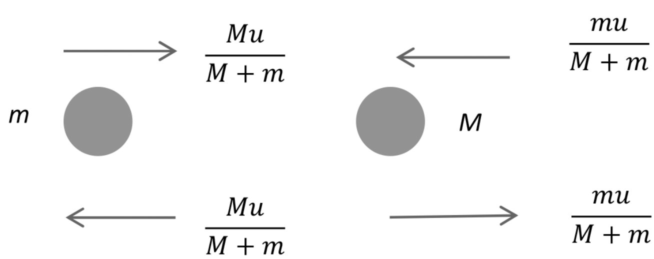


Figure 6.3: The collision of Figure 6.1 from the CM frame moving to the right with speed $w = mu/(M + m)$ in the laboratory frame. In the CM frame to conserve momentum, the particle velocities are reversed. To obtain the laboratory frame speeds add w (vectorially) to each

There are several ways to obtain the velocities in the CM frame. One way (we give an alternative in Section 6.5) is to note that if the particles are separated by a distance x, then the distances to the CM are $Mx/(M+m)$ and $mx/(M+m)$ and hence the speeds are $M\dot{x}/(M+m) = Mu/(M+m)$ and $mu/(M+m)$ toward the CM. During collision, these speeds are reversed in direction. Since the mass M was initially at rest in the laboratory frame, to get the speeds in this frame, we add $mu/(M+m)$ to each having regard to direction. Thus, the laboratory frame speeds after the collision are

$$V = \frac{mu}{M+m} + \frac{mu}{M+m} = \frac{2mu}{M+m},$$

for the mass M, and

$$v = -\frac{Mu}{M+m} + \frac{mu}{M+m} = \frac{m-M}{M+m}u,$$

for the mass m.

To give us confidence in dealing with collisions, let us derive this result in a less elegant but more straightforward way. The general rule is to write down the equations for conservation of momentum and conservation of energy. For the collision depicted in Figure 6.1, we have for conservation of momentum in the laboratory frame:

$$mu = mv + MV, \tag{6.2}$$

and for conservation of energy,

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + \frac{1}{2}MV^2. \tag{6.3}$$

Note that we measure all velocities in the same direction and let the algebra take care of the actual signs. In principle, both bodies could continue to move in the same direction after collision. We have to solve these equations for v and V. There is a neat trick that reduces the algebra. We write Equations (6.2) and (6.3) as

$$m(u-v) = MV, \tag{6.4}$$

and

$$mu^2 - mv^2 = m(u+v)(u-v) = MV^2. \tag{6.5}$$

Dividing Equation (6.5) by (6.4), we get

$$u + v = V.$$

Substituting back for V into (6.4) gives

$$v = \frac{m - M}{m + M} u, \tag{6.6}$$

and hence

$$V = \frac{2m}{m + M} u. \tag{6.7}$$

We see that if $M > m$, the direction of motion of the incoming particle is reversed.

6.5 CENTER OF MOMENTUM FRAME

Collisions are often easier to analyze in a frame in which the total momentum is zero (the center of momentum frame). Note that in this frame, both the initial and final momenta must be zero since momentum is conserved in collisions as collisions involve no external forces on the system. So suppose we have a body mass M moving at speed U and a mass m moving at u. To bring the total momentum to zero, we view the collision from a frame moving with speed w such that

$$(U - w)M + (u - w)m = 0. \tag{6.8}$$

Note that by specifying all the velocities in the same $+x$-direction the signs will look after themselves. Solving (6.8) for w, we find

$$w = \frac{MU + mu}{M + m}. \tag{6.9}$$

This is the speed of the center of momentum frame viewed from the laboratory (rest) frame.

In the center of momentum frame, before collision, the mass M moves with speed

$$U_{\mathrm{CM}} = U - w = \frac{m(U - u)}{M + m}, \tag{6.10}$$

and the mass m with speed

$$u_{CM} = u - w = \frac{M(u-U)}{M+m}. \tag{6.11}$$

Let the speed after collision be V_{CM} and v_{CM}, respectively, in the CM frame. Then conservation of momentum implies

$$MV_{CM} + mv_{CM} = 0. \tag{6.12}$$

If the collision is elastic, then conservation of energy gives

$$\begin{aligned} MV_{CM}^2 + mv_{CM}^2 = MU_{CM}^2 + mu_{CM}^2 &= M\left[\frac{m(U-u)}{M+m}\right]^2 + m\left[\frac{M(u-U)}{M+m}\right]^2 \\ &= \frac{mM}{M+m}(U-u)^2. \end{aligned} \tag{6.13}$$

Thus, using (6.12) to eliminate V_{CM},

$$\left(1+\frac{m}{M}\right)mv_{CM}^2 = \frac{mM}{M+m}(U-u)^2,$$

or

$$v_{CM} = \pm\frac{M}{M+m}(U-u).$$

We take the positive sign, otherwise $v_{CM} = u_{CM}$, and the collision would have no effect. In the laboratory frame, the speed is

$$v = v_{CM} + w = \frac{2MU + (m-M)u}{M+m}. \tag{6.14}$$

If $M = m$, we recover $v = U$ as in our first example, and if $U = 0$, we recover $v = (m-M)u/(M+m)$ as in (6.6).

We have also for the mass M:

$$V_{CM} = \frac{m}{M+m}(u-U),$$

and

$$V = V_{\mathrm{CM}} + w = \frac{2mu + (M-m)U}{M+m}. \qquad (6.15)$$

6.6 INELASTIC COLLISIONS

Another interesting example is where the particles stick together after a collision (e.g., in a possible road traffic accident) or where they split apart (as in an explosion). In neither case can we conserve both energy and momentum. Since there are no external forces acting, it is momentum that must be conserved. From the point of view of the calculation, we replace the conservation of energy by a further condition on the final speeds.

For example, if the particles stick together, we have $V_{\mathrm{CM}} = v_{\mathrm{CM}} = 0$ in the centre of momentum frame. The loss in energy is equal to the initial energy and therefore, from Equations (6.10) and (6.11),

$$\Delta E = \frac{1}{2} m u_{\mathrm{CM}}^2 + \frac{1}{2} M U_{\mathrm{CM}}^2 = \frac{mM}{M+m}(U-u)^2.$$

In an explosion, we have $u = U = 0$, and

$$\frac{1}{2} m v^2 + \frac{1}{2} M V^2 = \Delta E.$$

Conservation of momentum gives $mv + MV = 0$, so

$$\Delta E = \frac{1}{2}\left(1 + \frac{M}{m}\right) M V^2,$$

or

$$V = \left[\frac{m}{M}\frac{2\Delta E}{(m+M)}\right]^{\frac{1}{2}}.$$

If $M + m$ is a constant (the mass of the bomb), then the larger fragment carries off less energy by a factor $(m/M)^{1/2}$. This means that the smaller fragments can do more damage (depending on how many share the energy).

6.7 THE PROBLEM

We can now look at our initial problem as a series of elastic collisions.

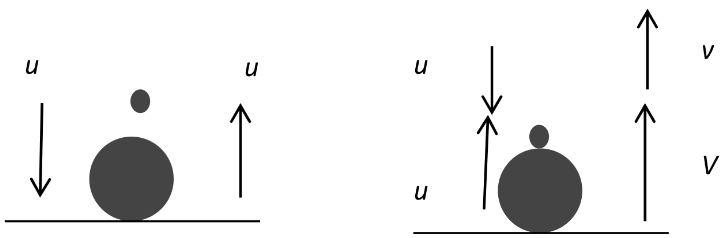


Figure 6.4: The figure on the left shows the balls at the point of initial impact (with the separation of the balls exaggerated); the figure on the right shows the subsequent collision

In the first stage, the big ball collides with the floor and has its motion reversed. We then have a collision between a small ball of mass m falling with speed u and a large ball of mass M rising with speed u.

Thus, from Equations (6.14) and (6.15) with $U = -u$

$$-v = \frac{\left(3 - \frac{m}{M}\right)}{1 + \frac{m}{M}} u \gg u,$$

if $m \ll M$, and

$$-V = \frac{1 - \frac{3m}{M}}{1 + \frac{m}{M}} u \sim u.$$

Thus, we expect the small ball to fly rapidly up once the big ball hits the floor. There is a transfer of energy from the larger to the smaller ball. This makes a surprisingly effective demonstration.

6.8 COLLISIONS IN TWO DIMENSIONS

Noncollinear collisions are a bit more difficult because we now have to conserve momentum in two directions. The general setup is shown in Figure 6.5

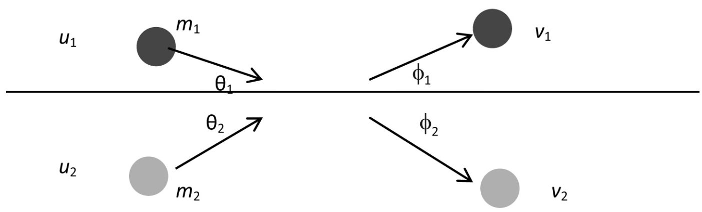


Figure 6.5: Bodies of masses m_1 and m_2 with speeds u_1 and u_2, respectively, traveling at angles θ_1 and θ_2 to the horizontal collide elastically. After the collision, the bodies move with the speeds and angles shown

We are given the initial parameters and have to find the final values after collision. Assume the collision is elastic. Then conservation of momentum (see Figure 6.5) yields

$$P_{\|} = m_1 u_1 \cos\theta_1 + m_2 u_2 \cos\theta_2 = m_1 v_1 \cos\phi_1 + m_2 v_2 \cos\phi_2 \quad (6.16a)$$

$$P_{\perp} = -m_1 u_1 \sin\theta_1 + m_2 u_2 \sin\theta_2 = m_1 v_1 \sin\phi_1 - m_2 v_2 \sin\phi_2 \quad (6.16b)$$

and conservation of energy gives

$$E = m_1 u_1^2 + m_2 u_2^2 = m_1 v_1^2 + m_2 v_2^2. \quad (6.17)$$

So we apparently have three equations for four unknowns. How do we fix this? Imagine repeating the experiment many times with the same initial angles and speeds: the result will be a variety of angles for the outgoing particles, because the angles will depend on just how glancing a blow they receive. So one of the emerging angles has to be given in order to determine the problem. Let this be ϕ_1. Then, eliminating ϕ_2 from (6.16a) and (6.16b),

$$(\mathrm{P}_{\|} - m_1 v_1 \cos\phi_1)^2 + (P_{\perp} - m_1 v_1 \sin\phi_1)^2 = m_2{}^2 v_2{}^2 = E - m_1 v_1{}^2,$$

which is a quadratic equation for v_1. We can then obtain v_2 and hence ϕ_2; but the general solution is far from illuminating. So let us take a specific example that does lead to a nice result.

We put $m_1 = m_2 = m$ and $u_1 = 0$. Then Equations (6.16a) and (6.16b) give

$$u_2 \cos\theta_2 = v_2 \cos\phi_2 + v_1 \cos\phi_1, \tag{6.18}$$

$$u_2 \sin\theta_2 = v_2 \sin\phi_1 - v_2 \sin\phi_2, \tag{6.19}$$

and (6.17) gives

$$u_2^2 = v_1^2 + v_2^2. \tag{6.20}$$

Thus, squaring and adding (6.18) and (6.19), we have

$$u_2^2 = v_1 v_2 (\cos\phi_1 \cos\phi_2 - \sin\phi_1 \sin\phi_2) + v_2^2 + v_1^2.$$

Using (6.20), we get

$$(\cos\phi_1 \cos\phi_2 - \sin\phi_1 \sin\phi_2) = \cos(\phi_1 + \phi_2) = 0,$$

or

$$\phi_1 + \phi_2 = \frac{\pi}{2}.$$

This is a useful result for snooker players. The cue ball and the target will move at 90° after collision.

6.9 COLLISION TIMESCALES

Since we can treat collisions by simply conserving momentum before and after the event, it is unnecessary to go into the details of what happens in a collision. However, the details can be important in clarifying what assumptions we are making in treating collisions in this way.

On impact, in a collision between two bodies, the bodies deform and this deformation is transmitted through the bodies with the speed of sound in the material of the bodies. The deformation of each body lasts on the order of the time it takes sound to cross the body and hence to transmit the force of impact to the body as a whole. If the force between the bodies is of order F and the time of impact of order Δt, the body is subject to an impulse of order $F\Delta t$. This is equal to the change in momentum (Section 6.3).

The impact of a golf club on the ball provides an interesting application. One might think that this is not a case of a momentum conserving collision because the golf club is being driven by the golfer. The ball and club head do not appear therefore to be an isolated system without external forces. However, the time of contact is about the sound crossing time of the golf ball which is about 0.5 ms. The time for the sound wave to travel up the shaft is about 1 ms. By the time, the golfer is aware of the impact the golf ball is well on its way. We can therefore treat the impact as conserving momentum.

6.10 ROCKET EQUATION

An interesting application of the conservation of momentum is the motion of a rocket. The expulsion of momentum in the exhaust gases is balanced by the gain in momentum of the rocket.

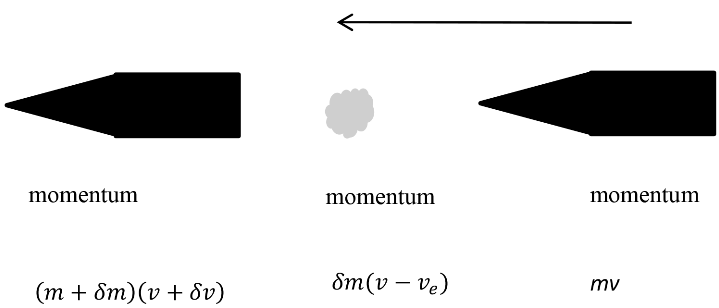


Figure 6.6: Derivation of the rocket equation; the rocket is moving from left to right

Suppose the rocket, mass m, ejects a mass δm with relative speed v_e. If the speed of the rocket is v before the gas is ejected and $v + \delta v$ after ejection conservation of momentum gives

$$mv = (m + \delta m)(v + \delta v) - \delta m(v - v_e).$$

Note that we write $m + \delta m$ for the mass of the rocket so that we can use the standard calculus approach that $dm/dt < 0$ corresponds to a loss of mass. Canceling terms gives

$$0 = m\delta v + \delta m v_e,$$

and hence, in the limit,

$$\frac{dv}{v_e} = -\frac{dm}{m}.$$

This integrates to

$$v_f = v_e \log_e \frac{m_i}{m_f}$$

for the final speed in terms of the initial and final masses. This is the *rocket equation*.

The rocket equation can be used to demonstrate the advantage of discarding the rocket casing along the way in multiple stage rockets to achieve higher speeds for a given payload.

6.11 CHAPTER SUMMARY

- In a system subject to no external forces momentum is conserved.
- The change in momentum of a system equals the impulse of the external forces $\int F\, dt$
- In an elastic collision momentum and energy are conserved.
- Collisions are often best looked at from the CM frame.
- The rocket equation gives the speed of a rocket in terms of the initial and current mass if the exhaust speed relative to the rocket is constant.

6.12 EXERCISES

1. Show that with the inclusion of gravity the rocket equation for near Earth flight becomes

$$v = v_e \log_e\left(\frac{m_i}{m_f}\right) - g\frac{m_i - m_f}{\dot{m}},$$

where $\dot{m}$ is the mass rate at which fuel is being burnt (assumed constant). It is possible to achieve a mass ration of $m_i/m_f \sim 10$ and a ratio of thrust to launch weight of ≤ 2. Show that this is insufficient to achieve escape speed from the Earth's surface in a single stage rocket.

2. For the special case of an n-stage rocket with identical stages and constant exhaust speed, find the final speed as a function of n and the fuel to mass ratio.

3. The HS-601 HP satellite uses XIPS ion thrust engines to perform north-south station keeping and to control roll and yaw. The xenon ions are ejected with a velocity of 33,600 m s^{-1}. The specific impulse (defined below) is 3400 s with a mass flow rate of 0.6 mg s^{-1}. Calculate the thrust and the power required from the solar array. The initial satellite mass is of order 1680 kg with about 380 kg of xenon fuel; if a single ion thrust motor were fired continuously, find the time to reach maximum speed and comment on the result.

The specific impulse of a rocket, I, is defined in terms of the thrust F (in Newtons) by $F = I\dot{m}g$.

4. Neutrons in a nuclear reactor are released from fissile material with energies of a few MeV and must be slowed down to speeds of a few km s^{-1} to maintain a chain reaction. They are slowed by elastic collisions with the nuclei of a moderator. How many head-on collisions would be required for a boron moderator? (In practice, the number is larger because glancing collisions are less effective in exchanging energy.)

5. Suppose two bodies have speeds u_1 and u_2 before a collision and speeds v_1 and v_2 afterward. The coefficient of restitution is defined by

$$e = \frac{v_2 - v_1}{u_2 - u_1}.$$

Regulations of the Association of Tennis Professionals (ATP) specify the height to which a ball must bounce on a hard surface. A ball dropped from a height of 254 cm will have a velocity of 7.06 m s^{-1} just before it hits the ground. According to the regulations, the tennis ball must then bounce to a height of between 135 cm and 147 cm, meaning the ball must have a velocity of between 5.14 m s^{-1} and 5.36 m s^{-1} as it leaves the ground. Calculate the range of acceptable values for the coefficient of restitution. How much energy is lost in the impact?

6. A bouncing ball dropped from a height h loses a fraction f of its energy on impact with the ground. What is the length of time before it comes to rest? (The time is finite even though the number of bounces is infinite.)

CHAPTER 7

Orbital Motion

Problem: The rings of Saturn and the accretion of material by a black hole are just two phenomena that depend on the "tidal" forces of gravity. Under what conditions does gravity disrupt an orbiting body?

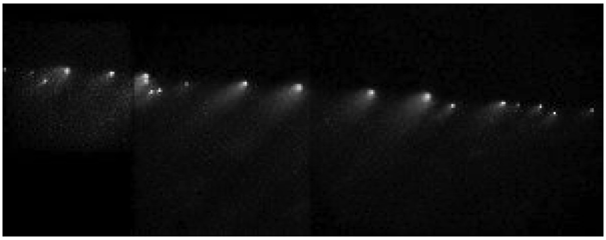


Image of the tidal disruption of Comet Shoemaker–Levy in the gravitational field of Jupiter (May 17, 1994) (NASA Image: STScI, *http://hubblesite.org/newscenter/newsdesk/archive/releases/1994/21/image/b*)

7.1 ANGULAR SPEED: GEOMETRIC APPROACH

Consider first a body moving uniformly round a circle; say a mass on the end of a string or a planet in a circular orbit about its parent sun. It is clumsy to describe this motion in Cartesian coordinates (x, y); it is much simpler to use polar coordinates (r, θ). The motion is then described by

$$\dot{r} = 0, \quad \dot{\theta} = \text{constant} = \omega \ (\text{say}),$$

and hence

$$r = \text{constant} \qquad \text{and} \qquad \theta = \omega t.$$

The quantity ω is called the *angular speed* of the body. In one revolution, θ changes by 2π so the time to complete one revolution is

$$T = \frac{2\pi}{\omega}.$$

We call T the *period* of the orbit. The number of orbits per unit time is the frequency $f = 1/T$.

We can see in Figure 7.1 that if in a time δt, the body moves through an angle $\delta\theta$ and it travels a distance $\delta s = r\delta\theta$. Its linear speed v is therefore

$$v = r\frac{\mathrm{d}\theta}{\mathrm{d}t} = r\omega.$$

Another way to see this for uniform motion is to consider that the body moves round the circle, a distance of $2\pi r$ in a time T, so

$$v = \frac{2\pi r}{T} = r\frac{2\pi}{T} = r\omega.$$

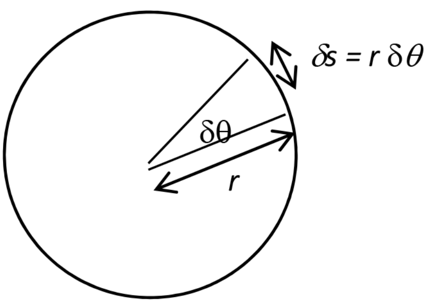


Figure 7.1: Circular motion

7.2 ANGULAR SPEED: ALGEBRAIC APPROACH

As an alternative to the geometric approach, we can describe the motion of a body in vectorial form. This will be useful in cases where the geometric picture is more complicated. Let $\hat{\mathbf{r}}$ be a unit vector in the radial direction, and let $\hat{\boldsymbol{\theta}}$ be a unit vector tangential to the circle (Figure 7.2)

From the figure, we have

$$\hat{\mathbf{r}} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j},$$

and

$$\hat{\boldsymbol{\theta}} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}.$$

Hence,

$$\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} = -\sin\theta\,\dot{\theta}\,\mathbf{i} + \cos\theta\,\dot{\theta}\,\mathbf{j}$$

$$= \omega\hat{\boldsymbol{\theta}}.$$

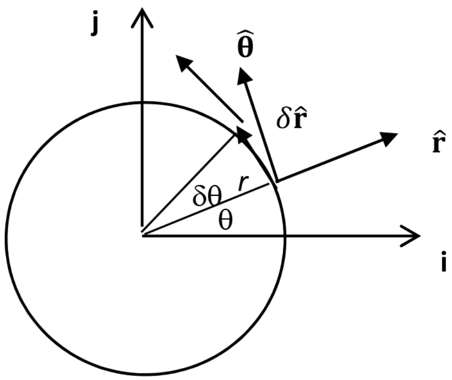


Figure 7.2: Angular speed in vector form

This agrees with the figure where we see that $\delta\hat{\mathbf{r}}$ is in the direction of $\hat{\boldsymbol{\theta}}$ and has magnitude $|\hat{\mathbf{r}}|\ \delta\theta = \delta\theta$ (for small enough $\delta\theta$). Similarly,

$$\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t} = -\cos\theta\,\dot{\theta}\,\mathbf{i} - \sin\theta\,\dot{\theta}\,\mathbf{j}$$

$$= -\omega\hat{\mathbf{r}}. \qquad (7.1)$$

Now consider

$$\mathbf{r} = r\,\hat{\mathbf{r}},$$

the position vector of the body. We have

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{r}\,\hat{\mathbf{r}} + r\omega\hat{\boldsymbol{\theta}}. \tag{7.2}$$

If $\dot{r} = 0$ (for circular motion), this tells us that the velocity of the body is $r\omega$ in the tangential direction. We also see, although this is obvious anyway, that if the body is not moving in a circle, the radial component of the velocity ($\hat{\boldsymbol{r}} \cdot \mathrm{d}\boldsymbol{r}/\mathrm{d}t$) is $\mathrm{d}r/\mathrm{d}t$.

7.3 ANGULAR VELOCITY AS A VECTOR

Any motion can be decomposed into a translation and a rotation about an axis. We can therefore think of the angular velocity as having a magnitude and direction if we assign the direction as the axis of rotation. We can then write the relation between angular velocity $\boldsymbol{\omega}$ and the linear velocity $\boldsymbol{v}$ of a body at position $\boldsymbol{r}$ as

$$\boldsymbol{v} = \boldsymbol{\omega} \wedge \boldsymbol{r}.$$

For example, if $\boldsymbol{\omega} = (0, 0, \omega)$, then a point in the x–y plane has velocity $(\omega y, -\omega x, 0)$.

7.4 ANGULAR ACCELERATION: GEOMETRIC APPROACH

Suppose a point moves with constant speed in a circle. The velocity vector of the body is tangential to the circle. In Figure (7.3), the velocity vectors from two neighboring points have been put into a triangle, so we can see clearly the change in velocity. There are two ways of calculating the change in radial speed. The tips of the velocity vectors $\boldsymbol{v}$ and $\boldsymbol{v} + \delta\boldsymbol{v}$ lie on a circle because the lengths of both sides of the triangle represent the same speed v. This triangle therefore is exactly like the one in Figure 7.2, where we compared two points on a circle.

So, just as in Section 7.1, where we had $\delta s = r\,\delta\theta$, here we have $\delta v = v\,\delta\theta$. Dividing by δt, we get that the acceleration $\mathrm{d}v/\mathrm{d}t$ is given by

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}\theta}{\mathrm{d}t} = v\omega = r\omega^2 = \frac{v^2}{r}. \tag{7.3}$$

Alternatively, the change is velocity is in the radial direction and hence perpendicular to the tangent. From the triangle on the right of Figure (7.3), the acceleration is

$$\frac{\delta v}{\delta t} = \frac{v\sin\delta\theta}{\delta t} = v\frac{\delta\theta}{\delta t}.$$

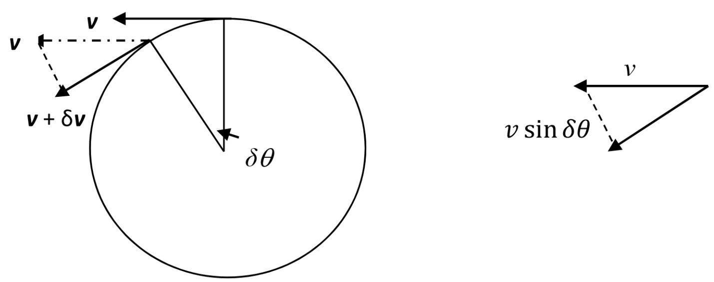


Figure 7.3: Angular acceleration

The various equivalent forms in Equation (7.3) are obtained by using $v = r\omega$ and they are all useful forms to remember. The expression $r\omega^2$ for the acceleration is just what we might guess on dimensional grounds: $r\omega^2$ is the only way, up to a constant, that we can make an acceleration from a length r and an angular speed ω per unit time.

Note that the component of acceleration tangential to the circle is zero because the speed of the body is constant. The direction of the acceleration is therefore radial, toward the center of the circle; we call this a centripetal acceleration.

7.5 ANGULAR ACCELERATION: ALGEBRAIC APPROACH

Continuing with the vectorial approach, we have from (7.2)

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{r}\,\hat{\mathrm{r}} + r\omega\,\hat{\boldsymbol{\theta}}.$$

So differentiating again, we get

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} + \dot{r}\omega\hat{\boldsymbol{\theta}} + r\dot{\omega}\hat{\boldsymbol{\theta}} + r\omega\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t}$$

$$= (\ddot{r} - r\omega^2)\hat{\mathbf{r}} + (\mathbf{2}\boldsymbol{\omega}\dot{\boldsymbol{r}} + \boldsymbol{r}\dot{\boldsymbol{\omega}})\hat{\boldsymbol{\theta}}. \tag{7.4}$$

If the motion is circular at constant speed, this tells us that the radial component of acceleration is just $-r\omega^2$, where the minus sign shows that it is toward the center. As a bonus, we get the acceleration for a general planar motion in polar coordinates: for the radial component of the acceleration (taking the scalar product of (7.4) with $\hat{\boldsymbol{r}}$) from (7.4)

$$a_r = \left(\ddot{r} - r\omega^2\right); \tag{7.5}$$

and for the tangential acceleration,

$$a_\theta = 2\omega\dot{r} + r\dot{\omega} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\omega\right), \tag{7.6}$$

where $\omega = \mathrm{d}\theta/\mathrm{d}t$.

7.6 ANGULAR MOMENTUM

Consider a particle of mass m moving in a plane subject to no nonradial forces. Then from (7.6),

$$m\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\omega\right) = 0,$$

and therefore,

$$mr^2\omega = \text{constant.}$$

Furthermore, the direction of the angular velocity will remain constant, normal to the plane. In contrast, the linear momentum $m\boldsymbol{v}$ is not constant in direction even if $|\boldsymbol{v}| = r\omega$ is constant in magnitude. We therefore define a new quantity, the *angular momentum*, by

$$\boldsymbol{h} = m\boldsymbol{r} \wedge \boldsymbol{v} = m\boldsymbol{r} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{r}) = mr^2\boldsymbol{\omega} - m\boldsymbol{r}(\boldsymbol{r} \cdot \boldsymbol{\omega}). \tag{7.7}$$

For the motion of a body in a plane, with no nonradial forces, $\boldsymbol{r}$ and $\boldsymbol{\omega}$ are perpendicular so $\boldsymbol{r}.\boldsymbol{\omega} = 0$ and from (7.7), the angular momentum vector $\boldsymbol{h}$ is in the direction of the angular velocity and is constant in magnitude and direction.

We shall generalize this from particles to extended bodies in Chapter 9.

7.7 CIRCULAR MOTION: DYNAMICS

For the general motion of a body in three dimensions, we have seen that Newton's second law can be written in vector form as

$$m\ddot{\boldsymbol{x}} = \boldsymbol{F}.$$

This is not, however, the most convenient way to deal with motion under forces directed toward a central fixed point. Rather we use the polar form for the acceleration derived in Section 7.5. Then

$$m\left(\ddot{r} - r\omega^2\right) = F_r \tag{7.8}$$

from (7.5) and

$$m\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\omega\right) = F_\theta. \tag{7.9}$$

from (7.6). For a radial force $F_\theta = 0$, so $r^2\omega$ = constant. Notice that this does not imply that the motion is circular (i.e., that r is a constant) because the angular speed might be changing. How is that possible? A radial force can act to change the speed of the body if the motion is not perpendicular to the radius. However, for the moment, we shall restrict ourselves to circular motion. Then both r and ω are constant for a given body. The only point at issue is how the angular speed in these circumstances depends on the radius of the circular orbit. In general, if $\ddot{r} = 0$, from (7.1)

$$\omega = \left(\frac{-F_r}{mr}\right)^{\frac{1}{2}}.$$

If F_r = constant, then

$$\omega \propto r^{-\frac{1}{2}},$$

and

$$v \propto r^{\frac{1}{2}},$$

so the angular speed decreases with r and the linear speed increases.

A more interesting example is $Fr = kr$ with k a constant. Then ω is constant, so two bodies at different radii from the center will remain along the same radius vector that they start from. This is the force law between quarks in hadrons (strongly interacting particles) such as the proton. If we assume that the mass of the proton is concentrated in the quarks (which is not really true, but will do for this exercise) and if we assume the quarks orbit at about the speed of light (which is in fact a good first approximation), then we can work out the radius R of a proton from its angular momentum. We have

$$R\omega = c$$

from the assumption that the quarks move at light speed. The angular momentum of a proton is $\frac{1}{2}\hbar$, where $\hbar = h/2\pi$ and h is Planck's constant.

Thus,

$$mRc = \frac{1}{2}\hbar,$$

from which (to order of magnitude)

$$R \sim \frac{\hbar}{mc} \sim 10^{-15}\text{m},$$

which is around the right value.

Another example is a circular orbit with constant angular momentum. If $mr^2\omega$ = constant, then $F = mr\omega^2 \propto 1/r^3$. This is the (fictitious!) force law implied by the tractor beam of the Starship Enterprise.

7.8 PARTICLE IN A MAGNETIC FIELD

An example of circular motion with a more complicated force law is an electrically charged particle, charge e, in a magnetic field. The force is orthogonal to the field, $\boldsymbol{B}$, and the velocity $\boldsymbol{v}$ of the particle and has a magnitude $vB\sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{B}$. The equation of motion for a circular orbit in a plane perpendicular ($\theta = \pi/2$) to a constant field is

$$m\frac{v^2}{r} = eBv.$$

Thus, $v \propto r$ and ω is constant, independent of r. The angular frequency $\omega = eB/m$ is called the *Larmour frequency*. Since the period is independent of r, a fixed frequency voltage can keep particles accelerating in circular orbits. This is the principle of the cyclotron.

It is interesting to work out the radius of the orbit of a particle moving with speed c (or, for pedantic readers, speed close to c). This is

$$r = \frac{mc}{eB}.$$

For an electron in the Earth's magnetic field, $r \sim 10$ m. This suggests, rather remarkably, that we could accelerate electrons to close to the speed of light in desktop machines with magnetic field strengths only a few orders of magnitude greater than that of the Earth. The fallacy (unfortunately) is that relativity makes an enormous difference as the particles approach the speed of light.

7.9 CENTRIFUGAL FORCE

Newton's laws hold in an inertial frame. In such frames, a body is held in a circular orbit as a result of a force toward the center that produces an acceleration toward the center. For example, the attentive parent insists that the child on the roundabout should hold on tight, because the tension in her arms will provide the necessary force to produce the required acceleration toward the center. If the child lets go, she is not hurled off from the roundabout so much as that she continues in the tangential motion required by Newton's first law. But what about the child's view from the roundabout?

From the child's frame of reference, the roundabout is at rest and the world is going round. Why does the child have to hang on in a system in which she is at rest? The answer is that Newton's laws do not hold in a rotating frame, because such a frame is not an inertial one. We can derive the correct equations of motion in a noninertial frame of reference only by starting in an inertial frame and making a transformation. We give the complete picture in the next section, but we already know what the answer must come out to be if Newtonian mechanics is to be consistent. In a frame of reference rotating with a body at angular speed ω relative to an inertial frame, the equation of motion is

$$F_r - mr\omega^2 = 0 = ma_r,$$

where a_r is the radial acceleration. So the radial acceleration in the rotating frame of reference is indeed zero, but a new term has appeared on the left to oppose the tension in the child's arms. This is called an *inertial force*. This force (and other such forces) appears whenever we view the world from a noninertial frame. (Note the slightly confusing nomenclature: inertial forces appear in noninertial frames.)

In this particular situation, of constant rotation about a fixed axis, we have a name for the inertial force: it is called *centrifugal force*. We see that the centrifugal force is in the opposite direction to the force holding the body in orbit, hence outward. The important fact is that centrifugal forces can only arise in noninertial frames. In inertial frames, there are only "real" forces and centripetal (toward the center) accelerations.

Example: Astronauts are trained to withstand high g-forces by being swung round lying on a rotating arm. The maximum centrifugal acceleration a pilot can tolerate without blacking out if he lies with feet outward is about 8g. Assuming the centrifuge arm is about 3 m what is the angular speed?

We identify the most useful form of the radial acceleration here as $r\omega^2$ and set it equal to 8g. Then

$$\omega = \left(\frac{8g}{r}\right)^{\frac{1}{2}} \sim 5 \text{ rad s}^{-1},$$

or a period of about a second for one rotation!

7.10 ROTATING FRAMES

In three dimensions, inertial forces are more complicated. Let a body have position vector $\boldsymbol{r}'$ in an inertial frame. Let us consider a frame rotating with angular velocity $\boldsymbol{\omega}$ relative to the inertial frame (we can always add translation later). Let the position vector of the body in this noninertial frame be $\boldsymbol{r}$. Then the velocity of the body relative to the inertial frame is composed of two components: the speed in the rotating frame and the speed of the rotating frame as seen from the inertial frame. Thus, we can write

$$\frac{\mathrm{d}\boldsymbol{r}'}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \boldsymbol{\omega} \wedge \boldsymbol{r},$$

or

$$\frac{\mathrm{d}\boldsymbol{r}'}{\mathrm{d}t} = \left(\frac{\mathrm{d}}{\mathrm{d}t} + \boldsymbol{\omega} \wedge\right)\boldsymbol{r}.$$

We see that the rate of change in the rotating frame is obtained by applying the operator $(\mathrm{d}/\mathrm{d}t + \boldsymbol{\omega}\wedge)$ to a vector. Thus, to obtain the acceleration in the rotating frame, we apply this operator to the velocity:

$$\frac{\mathrm{d}^2\boldsymbol{r}'}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}}{\mathrm{d}t} + \boldsymbol{\omega} \wedge\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} + \boldsymbol{\omega} \wedge\right)\boldsymbol{r}$$

$$= \frac{d^2\boldsymbol{r}}{dt^2} + 2\boldsymbol{\omega} \wedge \frac{d\boldsymbol{r}}{dt} + \boldsymbol{\omega} \wedge \boldsymbol{\omega} \wedge \boldsymbol{r} + \frac{d\boldsymbol{\omega}}{dt} \wedge \boldsymbol{r}.$$

If the rotation is steady, we can ignore the last term.

If we apply Newton's law $F = m\ddot{\mathbf{r}}'$ in the inertial frame, we have, in the rotating frame,

$$\boldsymbol{F} - 2m\boldsymbol{\omega} \wedge \frac{d\boldsymbol{r}}{dt} - m\boldsymbol{\omega} \wedge \boldsymbol{\omega} \wedge \boldsymbol{r} - m\frac{d\boldsymbol{\omega}}{dt} \wedge \boldsymbol{r} = m\frac{d^2\boldsymbol{r}}{dt^2}.$$

The term $-m\boldsymbol{\omega}\wedge\boldsymbol{\omega}\wedge\boldsymbol{r}$ is called the centrifugal force and the term $-2m\boldsymbol{\omega}\wedge d\boldsymbol{r}/dt$, or $-2m\boldsymbol{\omega}\wedge\boldsymbol{v}$, is called the *Coriolis force*. Note, once more, that these forces appear only in a rotating frame. So, for example, a body observed from a rotating Earth will be seen to experience these forces.

As a historical example (known as Newton's bucket), imagine you have a bucket of water supposedly at rest in front of you. The water surface is flat (apart from the very minor curvature due to surface tension at the sides of the bucket. If you now spin the bucket rapidly in a turntable, the water surface will adopt a parabolic form. An observer spinning round with the bucket will see a bucket at rest and a parabolic surface. The deformation of the water surface for no apparent reason indicates to this observer that they are not in an inertial frame.

It is not quite true that the surface of my stationary bucket is flat: if you could measure it arbitrarily accurately, you should find that it is slightly curved. This would tell me that the Earth is spinning and is therefore not an inertial frame.

If $\boldsymbol{\omega}$ is perpendicular to both $\boldsymbol{r}$ and $\boldsymbol{v}$, then the acceleration reduces to

$$\frac{d^2\boldsymbol{r}'}{dt^2} = \frac{d^2\boldsymbol{r}}{dt^2} - \omega^2\boldsymbol{r},$$

which is the more familiar form of the centrifugal acceleration

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Example: What is the ratio of the effective acceleration due to gravity at the equator and at the poles as a result of the Earth's rotation?

We have

$$\frac{g - R\omega^2}{g} = 1 - \frac{\left[6370 \times 10^3 \times \left(\frac{2\pi}{24 \times 3600}\right)^2\right]}{9.81}$$

$$= 0.997$$

7.11 GRAVITY

The most important central force on a large scale is that of gravity. Gravity is a relatively very weak force compared to the other fundamental forces (electromagnetism and the weak nuclear and strong forces). It takes the whole mass of the Earth to produce the modest acceleration of 9.81 m s^{-2} at its surface. But because it is long range and because there is no cancellation with antigravity, it usually dominates on macroscopic scales.

Newton's law of gravitation gives the force between two particles of masses M_1 and M_2 separated by a distance r as

$$F = G\frac{M_1 M_2}{r^2},$$

where $G = 6.67 \times 10^{-11}$ N kg^{-2} m^2 is *Newton's constant*. The force is attractive and acts along the line joining the two particles. In vector form, we can express it as:

$$\boldsymbol{F} = -G\frac{M_1 M_2}{r^3}\boldsymbol{r},$$

or

$$\boldsymbol{F} = -G\frac{M_1 M_2}{r^2}\hat{\boldsymbol{r}},$$

where $\hat{r}$ is a unit vector.

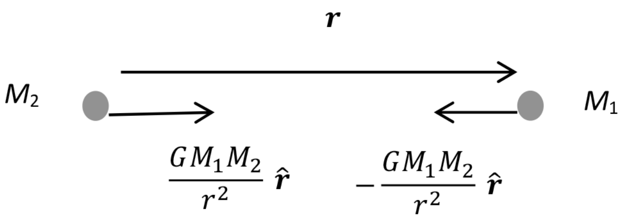


Figure 7.4: Gravitational action and reaction

Because gravity is a relatively weak force (the Earth produces an acceleration of only 1*g*) measuring *G* is quite difficult and its value is one of the least precisely known of the fundamental constants. Of course, if we take the mass of the Earth as known, then we can calculate *G* from the local acceleration due to gravity *g*, but this is circular because we need *G* to find the mass of the Earth. The value of *G* is determined using experimental setups that measure the force between two large masses, essentially sophisticated versions of that in Figure 7.4.

Example: As an example of the relative weakness of gravity let us estimate the gravitational attraction between Romeo and Juliet. Taking them to be point masses separated by 0.5 m, and allowing Juliet to be a little lighter than Romeo at 60 kg compared to 70 kg, gives a force of around 10^{-6} N. We can visualize how small this is if we look at the mutual potential energy (Equation (13.1) below): $GM_{\text{Romeo}}M_{\text{Juliet}}/0.5\,\text{m}$ is about 5×10^{-7} J. Assuming a watch battery lasts 5 years, this is sufficient to power a watch for less than a second.

7.12 EXTENDED BODIES

In the previous section, we have stated Newton's law of gravity for point particles and assumed it can be applied to extended bodies. This requires some justification (which was one of the difficulties that held up Newton for some time). For a spherical body (with a spherically symmetric mass distribution), the justification can be obtained by explicit integration. The setup is shown in Figure 7.5.

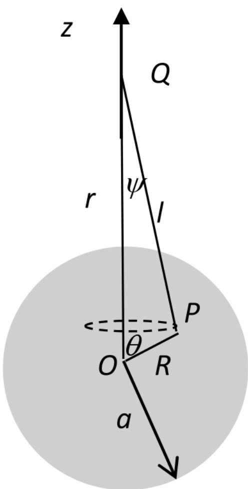


Figure 7.5: The gravitational force due to a uniform sphere

We evaluate the force on a point mass m at the point Q due to the spherical distribution of mass, density $\rho(R)$ centered on a point O a distance r from Q. The net force will be in the direction QO by symmetry so we shall evaluate this component only. Consider the element of mass at the point P, specified by the polar coordinates θ and ϕ with OQ as the z-axis. The force along QP is

$$\mathrm{d}F = \frac{Gm\rho R^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi \,\mathrm{d}R}{l^2}.$$

The component in the z-direction is

$$\mathrm{d}F \cos\psi = \mathrm{d}F \frac{(r - R\cos\theta)}{l}.$$

Using the cosine rule, we have

$$l^2 = r^2 - 2rR\cos\theta + R^2.$$

Thus, our integral becomes

$$F_z = \iiint \frac{Gm\rho R^2 \sin\theta (r - R\cos\theta)\,\mathrm{d}\theta \,\mathrm{d}\phi \,\mathrm{d}R}{\left(r^2 - 2rR\cos\theta + R^2\right)^{\frac{3}{2}}}$$

over the mass sphere. The ϕ integration just gives a factor of 2π. To evaluate the remaining integrals, note that we can write

$$F_z = 2\pi Gm \frac{\partial}{\partial r} \iint \frac{\rho R^2 \mathrm{d}R\ \mathrm{d}(\cos\theta)}{\left(r^2 - 2rR\cos\theta + R^2\right)^{\frac{1}{2}}},$$

with the limits on cos θ being 1 and –1. Now

$$\int_{-1}^{1} \frac{\mathrm{d}(\cos\theta)}{\left(r^2 - 2rR\cos\theta + R^2\right)^{\frac{1}{2}}} = -\frac{1}{rR}\left[\left(r^2 - 2rR\cos\theta + R^2\right)^{\frac{1}{2}}\right]_{-1}^{1}$$

$$= -\frac{1}{rR}\left[(r-R) - (r+R)\right] = \frac{2}{r}.$$

So finally, we get

$$F_z = 4\pi Gm \frac{\partial}{\partial r}\frac{1}{r}\int \rho R^2 \mathrm{d}R = \frac{GmM}{r^2},$$

where M is the mass of the gravitating body. This is exactly the same as if the body were a point mass at its center.

The Earth is approximately spherical. So, as far as its gravitational attraction is concerned, we are justified in treating it to a first approximation as a point mass at its center.

A nonspherical body certainly does not behave like a point mass except approximately at large distances. This is more difficult to justify (see Section 7.23).

7.13 GRAVITATIONAL POTENTIAL AND POTENTIAL ENERGY

In Section 4.4, we defined potential energy at a point in two (equivalent) ways: as the work done by a conservative force in moving a mass from a fixed reference point to the point in question and as a function the gradient of which gives the force. For the inverse square law of gravity, applying this second definition, we write for a point mass m,

$$F = \frac{GMm}{r^2} = -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{GMm}{r}\right).$$

Thus, the gravitational potential energy of a point mass m in the field of a mass M is

$$E_{\mathrm{P}} = -\frac{GMm}{r}. \tag{7.10}$$

Incidentally, the fact that the gravitational force can be written as the gradient of a function proves that gravity is conservative.

It is often not convenient to continually refer to the potential energy of a particular mass m; rather we can refer to the potential energy of a unit mass. We call the potential energy of a unit mass the *gravitational potential* and write it as $\phi(\boldsymbol{r})$

Calculating the work done in bringing a unit mass from infinity to a point a radial distance r from the origin along a radial trajectory gives

$$\phi = \int_{\infty}^{r} \frac{GM}{r^2}\,\mathrm{d}r = -\frac{GM}{r}.$$

To find the potential energy of a distribution of N point masses M_i at points $\boldsymbol{r}_i$, we sum over the distribution:

$$\phi(r) = -G\sum_{i=1}^{N} \frac{M_i}{|r - r_i|}.$$

In the limit of a continuous distribution, each element contributes a mass $\rho\,\mathrm{d}V$, and the potential energy becomes

$$\phi(r) = -G\int \frac{\rho}{|r - r'|}\,\mathrm{d}V'.$$

This expression will be useful later when we compare a general extended body to a spherical mass.

7.14 ESCAPE SPEED

The total energy of a body of mass m at a distance r from a spherical body of mass M is

$$E = E_K + E_P = \frac{1}{2}mv^2 - G\frac{Mm}{r}. \tag{7.11}$$

If there are no other external forces on the body, then the energy is a constant which can be positive or negative (or zero). If $E < 0$, then the maximum value for r is

$$r_{max} = \frac{GMm}{E},$$

when $v = 0$. The two bodies are therefore *bound*. If $E > 0$, then $r_{max} \to +\infty$; the separation of the bodies can be arbitrarily large. We say the bodies are *unbound*.

The marginal case is $E = 0$. To achieve this, we must have

$$v = v_{esc} = \left(\frac{2GM}{r}\right)^{\frac{1}{2}}.$$

We call this the *escape speed*. It is the speed of a body just sufficient to take it to infinity, that is, to allow it to escape the parent body starting from a radial distance r from the center.

Example: One simple illustration of the weakness of gravity is afforded by the escape speed from Deimos, the smaller of the two Moons of Mars. Use the following data to show that a human athlete can jump off of its surface into space.

Data: Mass 1.48×10^{15} kg, radius 6.2 km.

The escape speed is

$$v_{esc} = \left(\frac{2\times 6.67\times 10^{-11}\times 1.48\times 10^{15}}{6.2\times 10^{3}}\right)^{\frac{1}{2}} = 5.6\,\mathrm{m\,s^{-1}}.$$

We can compare this to the takeoff speed of a good high jumper: $v = \sqrt{2gh} \sim 12\,\mathrm{m\,s^{-1}}$.

7.15 RADIAL INFALL

If the motion under gravity is purely radial, then we can obtain a complete picture of the motion. We have θ = constant, $\omega = 0$, so the radial equation of motion (Newton's second law) is

$$m\ddot{r} = -\frac{GMm}{r^2}. \tag{7.12}$$

There are several ways of integrating this equation exactly, which we shall come to in a moment, but first let us do some estimates. Suppose you are falling from a height of h above the Earth, say in an airplane in a nosedive (neglecting air resistance). How long is it before you hit the ground?

If the fall time is t, we can estimate the acceleration as h/t^2 (based on dimensions: [acceleration] = L/T^2). So

$$\frac{h}{t^2} \sim \frac{GM}{r^2}.$$

If h is 2 km (say), then

$$t \sim r\left(\frac{h}{GM}\right)^{\frac{1}{2}} \sim 6370 \times 10^3 \times \left(\frac{2000}{6.67 \times 10^{-11} \times 6 \times 10^{24}}\right)^{\frac{1}{2}} \sim 14\,\text{s}.$$

Notice that the time is independent of the mass of the falling body. In the absence of air resistance, all bodied fall with the same acceleration.

Example: Another example is the time it would take for the Sun to collapse if the interior pressure were removed. In this case, $h \sim R_\odot$ and we have $t \sim \left(R_\odot^3/GM_\odot\right)^{\frac{1}{2}} \sim 1600$ s! The fact that this time is much shorter than the age of the Earth (and hence of the Sun) tells us that the Sun must be supported by internal pressure.

Now suppose we need to integrate the equation of motion (7.12) to obtain a more detailed picture of radial infall. The first thing we do is to choose some dimensionless variables so that we do not clutter the solution with factors of G and M. Specifically, if we identify a convenient radial scale R, we can define new variables x and τ by

$$x = \frac{r}{R}, \tag{7.13}$$

$$\tau = \left(\frac{R^3}{GM}\right)^{-\frac{1}{2}} t. \tag{7.14}$$

Note that the scaling of t can be obtained from dimensional analysis but it also follows from the previous example, where we found $\left(R^3/GM\right)^{\frac{1}{2}}$ to be a timescale, or from Kepler's third law that we discuss below (Section 7.16). Using the new variables, (7.12) becomes

$$\frac{d^2x}{d\tau^2} = -\frac{1}{x^2}. \tag{7.15}$$

Let us denote $dx/d\tau$ by x'. Then we can integrate (7.15) by multiplying through by x':

$$x'x'' = \frac{d}{d\tau}\left(\frac{1}{2}x'^2\right) = -\frac{x'}{x^2} = \frac{d}{d\tau}\left(\frac{1}{x}\right),$$

and hence

$$x' = -\left(\frac{2}{x} + 2\varepsilon^2\right)^{\frac{1}{2}}, \tag{7.16}$$

where $\varepsilon^2 = \frac{1}{2}x'^2(0)$ is the (dimensionless) initial energy per unit mass as $x \to \infty$. Using ε rather than $x'(0)$ itself tidies up the factors of 2. Note that we take the negative square root in (7.16) to represent an infalling body. Continuing, we can integrate (7.16):

$$-\frac{1}{\sqrt{2}}\int \frac{x^{\frac{1}{2}}}{\left(1+\varepsilon^2 x\right)^{\frac{1}{2}}} dx = \tau - \tau(0). \tag{7.17}$$

The integral can be performed by substituting $x = \frac{1}{\varepsilon^2}\sinh^2\theta$, giving, eventually,

$$\tau - \tau(0) = -\frac{1}{\sqrt{2}}\varepsilon^{-3}\left[\varepsilon x^{\frac{1}{2}}(1+\varepsilon x)^{\frac{1}{2}} - \sinh^{-1}\left(\varepsilon x^{\frac{1}{2}}\right)\right], \qquad (7.18)$$

where $x = 0$ at $\tau = \tau(0)$. If the infalling body starts with zero energy at $x = +\infty$, then $\varepsilon = 0$. It is an interesting mathematical exercise to work out the limit of (7.18) as $\varepsilon \to 0$, but it is much easier to start again from (7.17)! The complexity of (7.18) illustrates the value of an approximate solution (Figure 7.6).

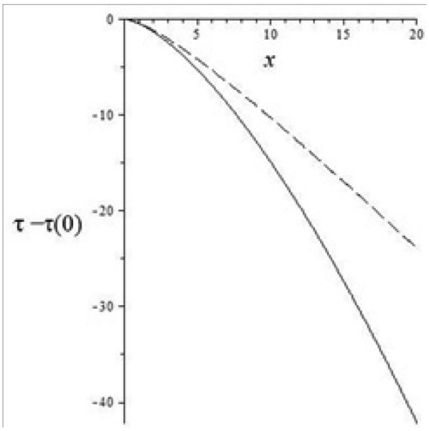


Figure 7.6: The solid line shows the approximate infall $\tau - \tau(0) = -\frac{\sqrt{2}}{3}x^{\frac{3}{2}}$ and the dashed curve the exact solution (7.18) for $\varepsilon^2 = 0.2$ with the condition that both go through the origin ($\tau(0) = 0$)

Example: Show that a body falling radially from rest to $r = 0$ at $t = 0$ satisfies

$$r = \left(\frac{3}{2}\right)^{\frac{2}{3}}(2GM)^{\frac{1}{3}}(-t)^{\frac{2}{3}},$$

for $t < 0$.

We integrate (7.16) with $\varepsilon = 0$. Thus,

$$\int x^{\frac{1}{2}} dx = -\sqrt{2} \int d\tau,$$

or

$$\frac{2}{3} x^{\frac{3}{2}} = -\sqrt{2}\tau.$$

Reinstating the physical variables from (7.13) and (7.14) gives the result.

7.16 CIRCULAR ORBITS

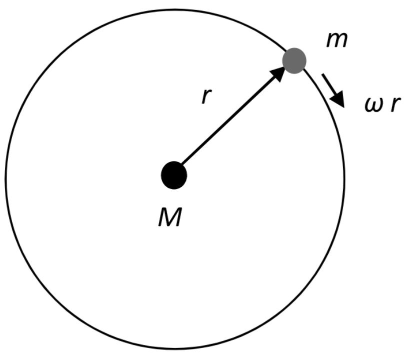


Figure 7.7: A circular orbit

Consider now a bound system. What are the stationary states? Consider first just two bodies of mass m and M with $M \gg m$. If we neglect the force of the smaller body on the larger body, then the latter will remain at rest in an inertial frame. We look for solutions in which r = constant, that is, circular motion. Using (7.3) for the acceleration, the equation of motion is

$$\frac{GMm}{r^2} = mr\omega^2$$

by Newton's second law.

Thus, the body remains in the fixed orbit if it has an angular speed:

$$\omega = \left(\frac{GM}{r^3}\right)^{\frac{1}{2}} \tag{7.19}$$

As usual, having obtained the formula we try to understand it. The equation tells us that only one specific angular speed will keep the body in orbit at a specific distance r. This is reasonable: if the body moves too quickly for the gravitational force that is exerted on it, it will fly off at a tangent. If it movers too slowly, it will fall toward the center. In either case, the radial distance cannot remain constant.

The formula also tells us that the angular speed falls off with distance. This means that the period $T = 2\pi/\omega$ increases with distance. In fact:

$$T = \left(\frac{4\pi^2}{GM}\right)^{\frac{1}{2}} r^{\frac{3}{2}}. \tag{7.20}$$

This is *Kepler's third law* for the special case of a circular orbit. It tells us in words that the square of the period is proportional to the cube of the orbital radius. This can be verified in the Solar System, given that the Sun is indeed much more massive than the planets and that the orbits of the planets are approximately circular.

Example: The Sun is 8 light minutes away, the Moon 1.25 light seconds. Using the ratio of the length of the month to the length of the year, estimate the ratio of the mass of the Sun to that of the Earth.

We have

$$\frac{T_\odot}{T_{\text{Moon}}} = \left(\frac{r_\odot}{r_{\text{Moon}}}\right)^{\frac{3}{2}} \left(\frac{M_{\text{Earth}}}{M_\odot}\right)^{\frac{1}{2}}.$$

Note that there is no need to convert the light travel times to SI units since we are dealing with ratios. (But the units need to be consistent, so we shall use light seconds for the orbital radii of both the Sun and Moon and months for their orbital periods.) Substituting numerical values, we get

$$\frac{M_{\text{Moon}}}{M_\odot} = \left(\frac{12}{1}\right)^2 \left(\frac{1.25}{480}\right)^3 \sim 2.5\times 10^{-6},$$

which is a slight underestimate.

7.17 VIRIAL THEOREM

From Equation (7.19), the kinetic energy of a body of mass m in a circular orbit of radius r about a body of mass M is

$$\frac{1}{2}mr^2\omega^2 = \frac{1}{2}\frac{GMm}{r}.$$

Thus, for a body in a circular orbit, using (7.9):

$$2E_K + E_P = 0. \tag{7.21}$$

This is a particular example of the *Virial Theorem* that asserts a relationship of this form in all cases of a system in a sufficiently symmetrical state of equilibrium. Note that the virial theorem gives an actual value for $2E_K + E_P$, in contrast to the energy Equation (7.10) which asserts only that $E = E_K + E_P$ is some constant <0. The virial theorem in the form (17.1) is not valid for a body in an elliptical orbit (which is where the symmetry requirement comes in). There is a more complicated form of the theorem for this case, but that is beyond the scope of this discussion.

7.18 CHANGING ORBITS

Here is an argument that sometimes seems paradoxical. If we are in a spacecraft in orbit and we burn the motor and increase the speed, the radius of the orbit will increase. If our speed were to increase above escape velocity, then we would escape from orbit. But the equation for orbital velocity is $v = (GM/r)^{1/2}$ so according to this as v increases r gets smaller not larger! What have we missed?

This makes it look as if formulae either do not work or only serve to complicate the issue. The problem is that a formula is valid only in the model for which it was derived. In general, the orbital velocity is not $(GM/r)^{1/2}$; it cannot be: quite obviously, the orbital velocity at any point can be whatever you want it to be by burning your rocket motors. Therefore, the formula can only hold under certain restricted circumstances. In fact, the formula gives the speed of a body in a *circular* orbit. If we burn our rocket motors, we shall no longer be in a circular orbit. In fact, to get from one circular orbit to another, we obviously have to depart from a circle. We shall look at the form of the more general orbit in the next section.

There is another apparent paradox involved here. Notice that in a circular orbit, the smaller the radius of the orbit the greater the speed. So we can make our rocket go faster by attempting to slow down. (Point the rocket motors in the direction of travel). However, we can do this only at the expense of losing height – in the same way that we can increase our speed by jumping off of a chair. There is no paradox!

7.19 ELLIPTICAL ORBITS

To investigate if there are more general orbits than circular, we need the general equation of motion of a body under a central inverse square law of force. In deriving the equations of motion in Section 7.7 (Equations (7.7) and (7.8)), we assumed that the orbit lay in a plane. To start with we would like to see if this is always the case. We need the general equations of motion in three dimensions, which means writing Newton's second law for an orbit about a central mass M as a vector equation:

$$\ddot{\boldsymbol{r}} = -\frac{GM}{r^3}\boldsymbol{r}.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{r}} \wedge \boldsymbol{r}) = \ddot{\boldsymbol{r}} \wedge \boldsymbol{r} + \dot{\boldsymbol{r}} \wedge \dot{\boldsymbol{r}} = -\frac{GM}{r^3}\boldsymbol{r} \wedge \boldsymbol{r} + 0 = 0.$$

Thus, the angular momentum $m\boldsymbol{r} \wedge \dot{\boldsymbol{r}}$ must be constant in both direction and magnitude and hence the motion must be in a plane. This means that we can use the equations of motion in polar coordinates we developed in Section 7.7.

For the angular motion, we have

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0,$$

and hence,

$$r^2\dot{\theta} = h = \text{constant}. \tag{7.22}$$

The quantity h is the conserved angular momentum per unit mass of the body. The radial equation is

$$\ddot{r} - r\dot{\theta}^2 = \frac{\mu}{r^2},$$

where $\mu = GM$. Clearly, we can eliminate θ from these equations to get an equation for r:

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{\mu}{r^2}.$$

It is not obvious how to set about solving this equation. In any case, it would give us r as a function of time, which is not what we want to determine the shape of the orbit: we need r as a function of θ. So let us first transform the derivatives with respect to time to derivatives with respect to θ. We have

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{r'h}{r^2},$$

where $r' = \dfrac{\mathrm{d}r}{\mathrm{d}\theta}$, and

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{r'h}{r^2}\right)\frac{\mathrm{d}\theta}{\mathrm{d}t} = \left(\frac{r''h}{r^2} - \frac{2r'^2h}{r^3}\right)\frac{h}{r^2},$$

so finally,

$$\frac{r''}{r^2} - \frac{2r'^2}{r^3} - \frac{1}{r} = -\frac{\mu}{h^2}. \tag{7.23}$$

This is no easier to solve directly. However, the presence of r'^2 suggests that we try the substitution $u = 1/r$: then

$$u' = -\frac{r'}{r^2},$$

and

$$u'' = -\frac{r''}{r^2} + \frac{2r'^2}{r^3}.$$

Thus, substituting for r'' and r' in terms of u'' and u' in (7.23), we arrive at the simple linear oscillator form:

$$u'' + u = \frac{\mu}{h^2}.$$

The fact that $u - \mu/h^2$ satisfies a harmonic oscillator equation tells us that the orbits are all periodic. Solving for u, the shape of the orbit is given by

$$u = \frac{\mu}{h^2} + k\cos\theta,$$

where k is a constant determined by the initial conditions (the value of u at $\theta = 0$). It is more convenient to write this as

$$u = \frac{1}{l}(1 - e\cos\theta), \tag{7.24}$$

where $l = h^2/\mu$ and $e \leq 1$ is some constant (unrelated to the base of natural logarithms) called the *eccentricity* of the orbit.

If we plot this curve, we find that it is some sort of oval: in fact, it is an *ellipse*. The approach to proving this depends on how we define an ellipse. Below and in the end-of-chapter exercises, we give a couple of relations between (7.24) and more usual definitions of an ellipse.

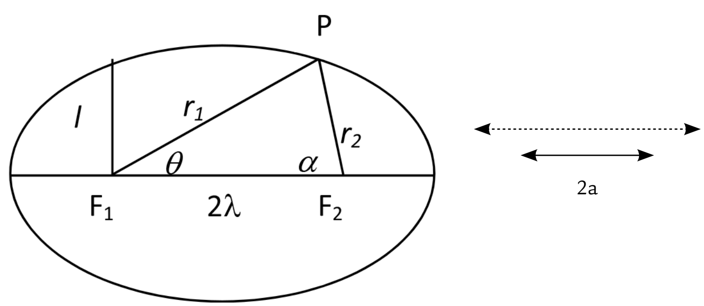


Figure 7.8: If $r_1 + r_2$ = constant the point P traces out an ellipse. F_1 and F_2 are the two foci of the ellipse

Consider first figure (7.21). There are two ways we can describe the figure by placing the central mass at either of the two focal points, F_1 and F_2. Thus, according to (7.23), the figure is equivalently described by

$$r_1(1 - e\cos\theta) = l,$$

and

$$r_2(1 - e\cos a) = l.$$

Adding these equations, we get

$$r_1 + r_2 = 2l + e(r_1\cos\theta + r_2\cos a)$$

But also, by simple trigonometry, the separation of the two foci is given by

$$r_1\cos\theta + r_2\cos a = \text{constant} = 2\lambda,$$

say. Thus,

$$r_1 + r_2 = 2l + e(r_1\cos\theta + r_2\cos a) = 2l + 2e\lambda = \text{constant}. \quad (7.25)$$

The constancy of the "length of string" between pegs at the two foci as the figure is traced out is one definition of an ellipse.

Note that the gravitating central body is at one focus, not at the geometric center.

7.20 PROPERTIES OF THE ELLIPSE

We can deduce certain properties of the orbit form the Equation (7.24). When $\theta = \pi/2$, we have $r = l$. This radius vector from the focus to the ellipse perpendicular to its major axis is called the *semilatus rectum*.

When $\theta = 0$, $r = l/(1 - e)$ and when $\theta = \pi$, $r = l/(1 + e)$, so the major axis, length $2a$ by definition, is

$$2a = \frac{l}{1-e} + \frac{l}{1+e} = \frac{2l}{1-e^2},$$

or

$$l = a\left(1 - e^2\right). \tag{7.26}$$

Considering the picture with $\theta = \pi$, $r = l/(1 + e)$, we now have

$$\lambda = a - \frac{l}{1+e} = ae \tag{7.27}$$

(using (7.26)) so, from (7.25)

$$r_1 + r_2 = 2l + 2e^2a = 2a. \tag{7.28}$$

If we now consider the triangle in Figure 7.9, by symmetry and using (7.25), (7.27), and (7.28)

$$r = \frac{1}{2}(r_1 + r_2) = l + e^2a = a.$$

Thus,

$$b^2 = a^2 - a^2e^2 = a^2\left(1 - e^2\right),$$

and

$$b = a\left(1 - e^2\right)^{\frac{1}{2}}. \tag{7.29}$$

This illustrates a very important point about planetary orbits. The largest value of e for solar system planets is about 0.1 for Mars. Thus, from (7.29), the difference in the lengths of the major and minor axes of the orbit of Mars is about 1%. To all intents and purposes, sketches of planetary orbits in the Solar System should look like circles.

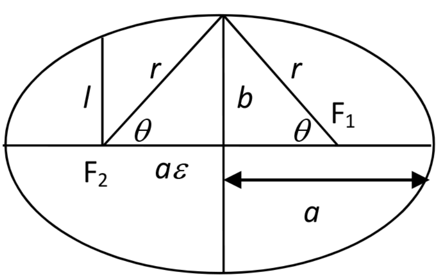


Figure 7.9: Parameters of the ellipse

This small departure form a circle is a consequence of the difference between the axes being of order e^2, the *ellipticity* of the ellipse. On the other hand, the displacement of the foci from the center is of order e, the *eccentricity* of the ellipse. The closest and furthest approaches of a planet to the Sun, situated at one focus, *are* significantly different.

The usual Cartesian form of the ellipse is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where the origin is at the center of the ellipse, not at a focus. It is a challenge to derive this from our definition; for the interested reader, some hints are given in one of the end-of-chapter questions.

Finally, in this section, we summarize the connection between the geometrical parameters of the ellipse and the physical parameters. Physically, the ellipse is defined by the angular momentum per unit mass h^2 and the energy per unit mass $\varepsilon = -GM/2a$. To derive the expression for energy consider the orbiting body at the position of periastron (closest approach to the parent star), $r = a(1 + e)$, $\theta = 0$. Then

$$\varepsilon = -\frac{GM}{r}+\frac{1}{2}r^2\omega^2 = -\frac{GM}{a(1+e)}+\frac{1}{2}a^2(1+e)^2\omega^2,$$

because the motion is orthogonal to the radius vector at this point. But

$$\omega a^2(1+e)^2 = h^2.$$

Substituting for ω and using $h^2 = GMl$ together with (7.26) gives (after some algebra)

$$\varepsilon = -\frac{GM}{2a}.$$

Since the energy is constant, we have also the so-called vis-viva ("energy of motion") equation

$$\frac{1}{2}v^2 - \frac{GM}{r} = -\frac{GM}{2a}.$$

Geometrically, the ellipse is defined by the semilatus rectum l and the eccentricity e with

$$l = \frac{h^2}{GM} \quad \text{and} \quad e^2 = 1 - \frac{h^2}{2\varepsilon}.$$

So, given the geometry of the orbit, we can work out the energy and angular momentum per unit mass, and given the energy and angular momentum of the orbit. we can work out the shape of the ellipse.

7.21 KEPLER'S LAWS

Prior to Newton's solution for the shape of an orbit under an inverse square law attractive force, Kepler had determined the properties of the orbit of the planet Mars through careful observation. He summarized his results in three laws:

1. The orbit is an ellipse with the Sun at one focus
2. The radius vector from the Sun to the planet sweeps out equal areas in equal times
3. The square of the orbital period is proportional to the cube of the semimajor axis

The third law relates, of course, not just to Mars but to the planetary system as a whole and to the moons of the giant planets.

We have just seen how to derive the first law (Section 7.19) if the central body is much more massive than the planet. For the second law, consider a circular orbit of radius r. The area of the circle swept out per unit time is the area of a triangle of base $r\omega$ and height r so $\dot{A} = \frac{1}{2}r\omega^2$. But $r^2\omega = h$, a constant and since r is constant so is $\dot{A}$. This is Kepler's second law for a cicular orbit. We have seen how to derive the third law for circular orbits in Section 7.16 (Equation 7.20).

The third law has to be modified to take account of the fact that the central body is not stationary but that two bodies orbit about their common CM. Let us calculate this effect.

The equations of motion for the two bodies with position vectors $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are

$$m_1 \ddot{\boldsymbol{R}}_1 = -\frac{Gm_1\, m_2}{R^2}\hat{\boldsymbol{R}}_1, \tag{7.30}$$

and

$$-m_2 \ddot{\boldsymbol{R}}_2 = -\frac{Gm_1\, m_2}{\boldsymbol{R}^2}\hat{\boldsymbol{R}}_2,$$

where $\boldsymbol{R} = \boldsymbol{R}_1 - \boldsymbol{R}_2$. Adding the equations tells us that the acceleration $\ddot{\boldsymbol{R}}$ is in the direction of $\boldsymbol{R}$ and hence that the bodies lie on opposite ends of a diameter. Since $(m_1 + m_2)\ddot{\boldsymbol{R}}_{\text{CM}} = m_1\ddot{\boldsymbol{R}}_1 + m_2\ddot{\boldsymbol{R}}_2 = 0$, the bodies revolve around their common CM. Furthermore, (7.30) gives

$$\ddot{\boldsymbol{R}}_1 = \frac{G(m_1 + m_2)\hat{\boldsymbol{R}}_1}{R_1^2},$$

so the orbiting bodies appear to be subject to the gravity of the sum of their masses. Thus, Kepler's third law (for bodies of mass m and M) becomes

$$T^2 = \frac{4\pi^2}{G(M + m)}R^3.$$

7.22 DERIVATION OF KEPLER'S LAWS FOR ELLIPTICAL ORBITS

For the general case of elliptical orbits, Kepler's laws can be derived from the Newtonian equations of motion in Section 7.19. We shall restrict our discussion to orbits about a fixed central mass. We have already seen that the first law is satisfied. The conservation of angular momentum (7.22) is equivalent to the second law just as in the case of circular orbits discussed in Section 7.21. For the third law, we start from an expression for the angular speed from (7.22) and (7.23):

$$\dot{\theta} = \frac{h}{l^2}(1+\varepsilon \cos\theta)^2. \tag{7.31}$$

Using $l = \frac{h^2}{\mu}$, we have $h/l^2 = l^{\frac{3}{2}}/(GM)^{\frac{1}{2}}$ so (7.31) gives

$$\frac{l^{\frac{3}{2}}}{(GM)^{\frac{1}{2}}}\int_0^{2\pi} \frac{d\theta}{(1+\varepsilon \cos\theta)^2} = \int_0^T dt = T.$$

Thus the period, T, is given by

$$T^2 = \frac{a^3}{GM} f(\varepsilon),$$

which is Kepler's third law.

7.23 EXTENDED BODIES: MULTIPOLE EXPANSION

We have seen that the gravity of a uniform spherical body is equivalent to a point mass at its center. For more complicated configurations of mass, we have to sum (or integrate) over the mass distribution to obtain the gravitational field. In these cases, it is easier to obtain the gravitational potential first (because it is a scalar) and then differentiate to find the magnitude and direction of the force.

Consider first two equal masses, m, separated by a distance d (Figure 7.10). The gravitational potential at r, denoted here by $\phi(r)$, is

$$\begin{aligned}\phi(r) &= -\frac{Gm}{r_1} - \frac{Gm}{r_2} \\ &= -\frac{Gm}{\left|\boldsymbol{r} + \frac{\boldsymbol{d}}{2}\right|} - \frac{Gm}{\left|\boldsymbol{r} - \frac{\boldsymbol{d}}{2}\right|}.\end{aligned}$$

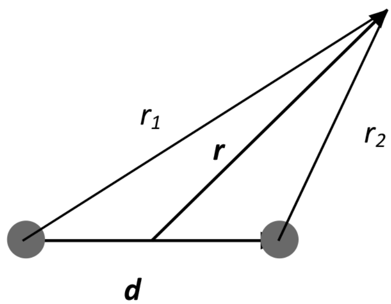


Figure 7.10: A gravitational quadrupole

If we look at this field far from the masses, so $r \gg d$, we can expand the denominators:

$$\frac{1}{\left|\boldsymbol{r}+\frac{\boldsymbol{d}}{2}\right|}=\frac{1}{(r^2+\frac{d^2}{4}+rd\cos\theta)^{\frac{1}{2}}}$$

$$=\frac{1}{r}+\frac{d}{2r^2}\cos\theta-\frac{d^2}{8r^3}+\frac{3}{8}\frac{d^2}{r^3}\cos^2\theta+\ldots.$$

Thus, to lowest nonzero order in d,

$$\phi=-\frac{2Gm}{r}-\frac{Gmd^2}{4r^3}(3\cos^2\theta-1).$$

We call this a gravitational *quadrupole*. There is an obvious possibility of extension to various octopoles (put two quadrupoles together in various ways) and higher moments. What would have been the lowest-order dipole term, linear in d/r^2, has canceled because we have taken the CM as the origin of coordinates.

For a continuous mass distribution, we have

$$\phi=-\int\frac{G\rho(r')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V', \tag{7.32}$$

which can also be expanded in multipole moments if $r \gg r'$. The first two terms are

$$\phi=-\frac{GM}{r}-\frac{Q}{r^3}(3\cos^2\theta-1)+\ldots,$$

where again we have assumed that the origin of coordinates in the CM to remove the dipole ($1/r^2$) term. Higher-order terms involve increasingly negative powers of r multiplied by various polynomial functions of θ called Legendre polynomials, $P_n(\theta)$, the expressions for which can be obtained from published tables.

7.24 THE POISSON EQUATION

From (7.32), we can derive a differential equation that must be satisfied by the Newtonian gravitational potential:

$$\nabla^2 \phi = -4\pi G\rho, \tag{7.33}$$

where, $\nabla^2 \phi \equiv \nabla.\nabla\phi$ (or div(grad ϕ)) and is, in Cartesian coordinates,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}.$$

Equation (7.33) is called Poisson's equation.

For the interested reader, we can derive (7.33) from (7.32) as follows. Surround the point $\boldsymbol{r} = \boldsymbol{r}'$ by a small sphere S_ε of radius ε and unit normal $\boldsymbol{n}$, so $\boldsymbol{r} - \boldsymbol{r}' = \varepsilon\boldsymbol{n}$. Surround the whole system by a large sphere outside the matter distribution, then we have

$$\nabla^2\phi = -G\int\rho(\boldsymbol{r}')\nabla.\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\,dV' = G\int\rho(\boldsymbol{r}')\nabla'.\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}dV'$$

$$= G\int_{S_\varepsilon}\rho(\boldsymbol{r}')\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}.\boldsymbol{n}\,dS',$$

where the first step follows because $\rho(\boldsymbol{r}')$ is not a function of $\boldsymbol{r}$; in the intermediate step, we have used the fact that the derivative of $|\boldsymbol{r} - \boldsymbol{r}'|$ with respect to $\boldsymbol{r}'$ is the negative of that with respect to $\boldsymbol{r}$; the last step follows from the divergence theorem (or Gauss's theorem) which relates a volume integral of the divergence of a vector to a surface integral, and we have neglected the contribution from the distant sphere. The remaining integral is over the small sphere surrounding the point $\boldsymbol{r} - \boldsymbol{r}'$. Since ρ is approximately constant in a small enough region, we can take it out of the integral and the expression becomes

$$\nabla^2\phi = G\rho\,(r)\int\nabla\left(\frac{1}{\varepsilon}\right).\boldsymbol{n}\varepsilon^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = -4\pi G\rho,$$

since $\nabla(1/\varepsilon) = -(1/\varepsilon^2)\boldsymbol{n}$.

We can now summarize Newtonian gravity in two equations:

1. Poisson's equation for the gravitational potential and
2. The appropriate form of Newton's second law for a gravitational force:

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -\nabla\phi. \tag{7.34}$$

Notice that the mass of the moving body does not appear in (7.34).

7.25 MOTION INSIDE MATTER: FALLING THROUGH THE EARTH

We can use Poisson's equation to determine the gravitational field inside a matter distribution, for example, in the interior of the Earth.

For a spherical system (where ϕ depends only on the radial coordinate r) Poisson's equation (7.33) can be shown to take the form

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\phi\right) = -4\pi G\rho. \tag{7.35}$$

To integrate this, let us take ρ to be constant. We now want the potential to vanish at $r = 0$ not at $r = \infty$ so we integrate from r to 0. Integrating (7.35) then gives

$$\phi = \frac{2}{3}\pi r^2 G\rho. \tag{7.36}$$

We could also obtain this result directly by assuming that the material external to radius r makes no contribution to the gravitational force at r. Then

$$F = -\frac{d\phi}{dr} = -\frac{GM(r)}{r^2},$$

where $M(r)$ is the mass within a radius r. Setting $M(r) = \frac{4}{3}\pi r^3 \rho$ (recalling that we have assumed ρ to be constant) and integrating gives us (7.36). Note that (7.37) is not the same as assuming only the total interior mass contributes to the gravitational potential ϕ: as can be seen from (7.36), ϕ is not equal to $-GM(r)/r$. (Why is what is true for the force not true for the potential? A spherical mass distribution acts as if from a central point only for an inverse square law; the potential does not satisfy an inverse square law.)

Thus, a body falling though a radial shaft through the Earth satisfies

$$\frac{d^2r}{dt^2} = -\frac{d\phi}{dr} = -\frac{4\pi}{3}Gr\rho.$$

This is the equation for SHM. We can use it to estimate the free-fall time through the Earth which is not a very practical exercise but an amusing piece of useless information. It is, however, instructive to try to guess the result before doing the calculation.

Estimating $\ddot{r}$ as R/T^2 on dimensional grounds as usual, and taking 5500 kg m^{-3} for the density of the Earth, we have

$$T \sim \left(\frac{4\pi}{3}G\rho\right)^{-\frac{1}{2}} \sim 814\,\text{s},$$

which is, perhaps surprisingly, independent of the radius of the Earth! Why is this?

The answer is that the period of SHM is independent of amplitude! On a larger planet, a body would have further to fall, but, if ρ is the same, the acceleration would be greater, and so would the speed over much of the fall. This compensates for the larger distance. So this turns out to be an interesting example of a case of exact SHM for large amplitudes. Perhaps equally surprising is the fact that the time is the same if we dig the shaft radially or along a chord. (See the end-of-chapter exercises.)

The exact solution to (7.38) is obtained by noting this is the equation of SHM with angular frequency $\omega = \sqrt{(4\pi G\rho/3)}$, so the period (in the model of constant density) is 2π larger than our estimate above.

7.26 TIDAL FORCES

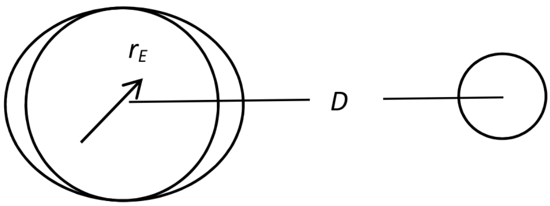


Figure 7.11: Tides raised on the Earth by the Moon

The force of the Moon on the Earth is greater on the side nearer the Moon than on the opposite side. On a spherical Earth, this differential force would affect sea level on both the side of the Earth facing the Moon and the side facing away, giving rise to two tides a day as the Earth spins underneath the bulges in the ocean. In practice, tides are affected by local geography, so two tides every 12 h is a very rough approximation.

To see the effect, consider the difference in gravitational acceleration at the Earth's center and at its surface due to the Moon, directly below the Moon:

$$\Delta g = \frac{Gm}{(D - r_E)^2} - \frac{Gm}{D^2}$$
$$\sim -\frac{2Gmr_E}{D^3},$$

where we have used the fact that $r_E \ll D$ to expand the denominator by the binomial theorem and to neglect terms of order $(r_E/D)^2$ and higher. Similarly, on the far side, we have

$$\Delta g = \frac{Gm}{D^2} - \frac{Gm}{(D + r_E)^2} \sim -\frac{2Gmr_E}{D^3}.$$

In both cases, gravity is weakened by the presence of the Moon. Putting in values gives $\Delta g{\sim}10^{-7}g$.

Example: We can estimate the height, h, of the tides by looking at the change in radius of the Earth that would give this change in gravitational acceleration. We have

$$\frac{GM}{(r_E + h)^2} = g - \Delta g,$$

or, approximately,

$$\frac{GM}{r_E^{\,2}} - \frac{GMh}{r_E^{\,3}} = g - \Delta g,$$

and hence

$$h = r_E \frac{\Delta g}{g} \sim 0.6\text{m}.$$

This simplified theory ignores the motion of the center of gravity of the Earth due to the Moon that causes the Earth to oscillate around its mean orbit. Since the seas are free to flow, they do not follow the oscillation in step which adds to the tidal amplitude. Actual tides also vary greatly round this value as a result of local geography.

7.27 SOLUTION OF THE PROBLEM: ROCHE LIMIT

We are now in a position to solve the original problem of the breakup of a star by a black hole. In fact, there is nothing special about a black hole as far as its gravity is concerned. The external gravitational field of a nonrotating hole is the same as that of a normal spherical body of the same mass, so the discussion would be the same for, say, a moon orbiting a planet.

A star will break up if the tidal forces on it exceed its own gravity. In the notation of Section 7.26 breakup occurs if

$$\Delta g \sim g.$$

Thus, the condition for a star of radius R approaching at a distance D is

$$\frac{2GMR}{D^3} \sim \frac{Gm}{R^2},$$

or

$$D = R\left(\frac{2M}{m}\right)^{\frac{1}{3}}$$

is the distance of closest approach. This is called the *Roche limit*. For a black hole and a star, we can scale this in terms of the Sun:

$$D = 7 \times 10^8 \left(\frac{R}{R_\odot}\right)\left(2\frac{M}{M_\odot}\frac{M_\odot}{m}\right)^{\frac{1}{3}} \text{ m}.$$

So for a star like the Sun approaching a black hole in the center of a galaxy, with typically a mass of $10^8 M_\odot$ and a radius of 3×10^{11} m, the distance of closest approach is around 4×10^{11} m or a few times the distance between the Earth and Sun.

7.28 WHAT IS GRAVITY?

Finally, we turn to a common problem at this point that occurs to students which is: what exactly is gravity? Students are in good company – the problem also occurred to Newton's contemporaries. What they wanted was a mechanical picture of how the Sun could affect the distant planets. Newton's answer was, in effect, that we could not say anything more about gravity than how it acted.

This is a key moment in the development of science. The fact is that only observation and experiment tells us what exists in the world not our commonsense notions or everyday experience. Gravity is not reducible to anything else, mechanical or otherwise, but is a separate component of the world. Since Newton's time we have discovered other forces that constitute these fundamental components, such as electricity and magnetism, which, despite a century of effort before

Maxwell, were also found not to be reducible to local mechanical interactions.

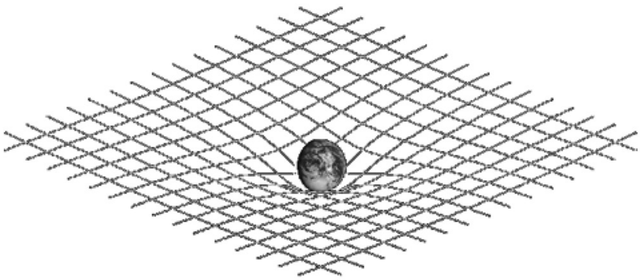


Figure 7.12: The rubber sheet analogy for the effect of a massive body (here the Earth) on the geometry of the space–time around it.
http://upload.wikimedia.org/wikipedia/commons/2/22/Spacetime_curvature.png

On the other hand, we may learn more about how to describe gravity and these other forces. In Einstein's general theory of relativity, gravity arises from the geometry of space–time as illustrated by the well-known rubber sheet analogy in Figure 7.12: the shape of space created by the Sun in its vicinity is what accounts for the inverse square law and keeps the planets their orbits.

Einstein's general theory of relativity has two parts: one, a theory of gravity and, two, the special theory of relativity. The first part, the theory of gravity, is still a theory of curved space and time even if we do not include special relativity to take account of strong gravitational fields and near light speeds. From this point of view (due to Cartan in 1922), Newtonian gravity is also a theory of the structure of space–time. Let us see briefly how this works.

The key is the *principle of equivalence* which is the main physical input into the theory. This tells us (in essence) that by performing experiments of restricted precision in a restricted region of space, we cannot distinguish between a situation in which we are observers at rest in a gravitational field and one in which we are observers in a laboratory (e.g., a spaceship) undergoing constant acceleration in the absence of gravity. From the first point of view, all bodies (subject to no nongravitational forces) fall with equal acceleration. One can express this by saying that the passive gravitational mass of a body m_P (the mass of a body that determines its response to a gravitational field) and its inertial mass m are equal: $ma = m_P g$ implies $a = g$, independent of m, if $m = m_P$. From the second point of view of a uniformly accelerated system, all bodies subject to no forces are seen to fall with the acceleration of the observer.

This has significant implications for Newtonian mechanics. It means that, once we admit the existence of gravity, we cannot identify the local inertial frames by any local experiments. This follows from what we have just said in the paragraph above, where we could not distinguish between an accelerated (i.e., noninertial) frame and on at rest (hence supposedly inertial). You might argue that we should just look out of the window and check whether the stars are at rest. But there is no causal connection between the stars and the local experiments: the falling bodies should not care whether the stars are rushing round or not. So a proper theory of gravity will not care whether we use inertial frames or not.

To obtain a theory of gravity, we now look at the same facts from a different viewpoint. The equivalence principle tells us equally that if we fall freely in a gravitational field, we cannot detect the effects of gravity by any local experiment. (All bodies will be weightless.) Thus, we know what physics in a gravitational field looks like from the point of view of a local freely falling observer: it looks exactly like physics without gravity! However, it is very inconvenient to keep hopping between local frames to do physics. (There is not really any technique for doing this.) We have to refer back to some global frame (it doesn't matter which). So to obtain our Newtonian theory of gravity, we simply transform from the local frame (where we know the physics) to the global reference frame (where we can do calculations). For the locally freely falling observer, the equation of motion of a body with coordinate ξ, subject to no nongravitational forces, is

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = 0. \tag{7.37}$$

For simplicity, we shall stick to motion in one dimension. We transform this equation to a general global frame of reference, $x = x(\xi)$. We have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}t}, \tag{7.38}$$

and

$$\frac{d^2x}{dt^2}=\frac{d^2x}{d\xi^2}\left(\frac{d\xi}{dt}\right)^2+\frac{dx}{d\xi}\frac{d^2\xi}{dt^2}=\frac{d^2x}{d\xi^2}\left(\frac{d\xi}{dx}\right)^2\left(\frac{dx}{dt}\right)^2, \qquad (7.39)$$

using (7.37) and (7.38) to obtain the final expression. Thus, rearranging (7.39), the equation of motion in the general x-frame can be written

$$\frac{d^2x}{dt^2}+\Gamma(x)\left(\frac{dx}{dt}\right)^2=0, \qquad (7.40)$$

where

$$\Gamma(x)=-\left(\frac{\partial\xi}{\partial x}\right)^2\frac{\partial^2 x}{\partial\xi^2}.$$

Equation (7.40) is the equation of motion of a body in a gravitational field in Newtonian physics, but now expressed in terms of a quantity, Γ (called an affine connection). The affine connection is a property of space–time because (although we cannot show it here) it allows us to identify vectors that are parallel to each other in an arbitrary curvilinear coordinate system (in which case they will not have coordinates that are proportional). Even in Newtonian theory, therefore, taking account of the equivalence principle, gravity appears, not as an additional force on the right of the equation of motion, but as a property of space–time on the left. We do not need to specify the infinity of local freely falling frames nor do we have to specify the inertial frames by some slight of hand. Rather we give Γ in *any* convenient frame and use (7.40) (or its generalization to three dimensions) to predict the path of a particle.

The second part of the problem is to relate Γ to the distribution of matter, since this is what determines the effect of gravity. To recover Newtonian physics, we impose

$$\Gamma=\nabla\phi\left(\equiv\frac{d\phi}{dx}\text{ in one space dimension}\right),$$

where ϕ is a solution of Poisson's equation. In fact, $\nabla^2\phi$ turns out to be the curvature of Newtonian space–time, so the Poisson equation relates the curvature of space–time to the distribution of matter, analogously to the relation in general relativity. On the other hand, a major difference between Newtonian theory and relativity is that the curvature in Newtonian physics does not affect the behavior of rods and clocks.

7.29 CHAPTER SUMMARY

- Angular speed ω and linear speed v a related by $v = r\omega$ or in vector form $\boldsymbol{v} = \boldsymbol{\omega} \wedge \boldsymbol{r}$
- The period of a circular orbit at constant angular speed ω is $T = 2\pi/\omega$
- The angular acceleration of a body moving in a circle of radius r is $r\omega^2 = v^2/r$ toward the center
- In general, the radial and tangential components of acceleration are $a_r = \left(\ddot{r} - r\omega^2\right)$,

$$a_\theta = 2\omega\dot{r} + r\dot{\omega} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\omega\right)$$

- Angular momentum is defined by $\boldsymbol{h} = m\boldsymbol{r} \wedge \boldsymbol{v}$
- For an observer in a rotating frame of reference $-m\boldsymbol{\omega}\wedge\boldsymbol{\omega}\wedge\boldsymbol{r}$ is called the centrifugal force and $-2m\boldsymbol{\omega}\wedge\boldsymbol{v}$ is called the *Coriolis force*
- *Newton's law of gravitation* gives the force between two particles of masses M_1 and M_2 separated by a distance r as $F = G\frac{M_1M_2}{r^2}$, where $G = 6.67 \times 10^{-11}$ N kg^{-2}m^2 is *Newton's constant*
- The *gravitational potential* of a mass distribution is

$$\phi(r) = -G\int \frac{\rho}{|r - r'|}\,\mathrm{d}V';$$

- The gravitational potential of a point mass is

$$\phi = \int_\infty^r \frac{GM}{r^2}\,\mathrm{d}r = -\frac{GM}{r}.$$

- Newton's law for a circular orbit is $\frac{GMm}{r^2} = mr\omega^2$
- The orbit of a body under an inverse square law of force is an ellipse given by $\frac{1}{r} = \frac{1}{l}(1 - e\cos\theta)$
- The Poisson equation for the gravitational potential of a distribution of matter, density ρ is $\nabla^2\phi = -4\pi G\rho$

7.30 EXERCISES

1. Tidal power extracts energy from the rotation of the Earth through its interaction with the Moon. In the process the potential energy of the Earth–Moon system is lost as the Moon escapes into space. It is interesting therefore to calculate the energy in the Earth–Moon system. Show that this is about 5% of the Earth's rotational energy.
2. The Sun is losing mass constantly via the conversion of nuclear fuel and a solar wind. How does the loss of mass affect the orbit of the Earth? What impact does this have on climate change?
3. Show that the orbit of a body under an inverse cube law of force is a logarithmic spiral.
4. In the text, we considered bound orbits. If the total energy of a two-body system is positive, show that the orbit of a test body (i.e., one with a mass much less than that of the central body) will be hyperbolic. Show further that the orbit will be hyperbolic also if the inverse square law of force were repulsive. (This meant that Rutherford's experiments on scattering alpha particles from nuclei did not allow him to distinguish whether the particles had like or unlike charges.)
5. Estimate the length of the seasons on Mars assuming they are defined by dividing the angle θ from the Sun to Mars into four quadrants of 90°.

CHAPTER 8

OSCILLATIONS

Question: How can a pendulum stabilize a building in high winds?

The *Tour Sans Fins* ("Tower without Ends") was a tower planned in La Defénse in Paris that has since been canceled. The spelling *Tour Sans Fins* (rather than the apparently correct French *fin*) comes from the idea that this tower had no ends, even if one were to look up or down at it, hence "ends" and not "end." The *Tour Sans Fins* was meant to be 400 m tall and would have been the tallest skyscraper in Europe. It would have used a large pendulum to damp any oscillations induced by high winds. Why would high winds induce oscillations? How can a coupled pendulum damp them?

8.1 RESONANCE

Consider an SHO, natural frequency ω_0, subject to a periodic force with angular frequency ω. The equation of motion of the oscillator is

$$\frac{d^2x}{dt^2} + \omega_0^2 x = f\cos\omega t. \tag{8.1}$$

Let us look for a solution of the form $x = A\cos(\omega t + \phi)$. Substituting into (8.1), we get

$$-A\omega^2 + A\omega_0^2 = f,$$

and therefore,

$$x = \frac{f}{\omega_0^2 - \omega^2}\cos(\omega t + \phi). \tag{8.2}$$

If we adjust the value of ω toward ω_0, by altering the frequency of the forcing term, we see that the amplitude becomes increasingly large. This is the phenomenon of *resonance*.

A simple example is a child's swing. A swing has a natural frequency with which, once started, it will rock to and fro. Giving the swing a push in synchrony with its natural frequency increases the amplitude. In fact, also any multiple of the period will amplify the motion. Eventually, the swing goes over the top, in which case, it will not behave as a swing – because the rope will slacken. More usually, at some stage, the resistance to motion increases and further pushing just compensates for this, so there is no further increase in the amplitude. On the other hand, pushing at random, or out of step, will sometimes decelerate the swing, sometimes accelerate it, with the result that there is only a small overall effect.

There are many examples of resonance on bridges as a result of people marching in step across them with a step that is some multiple of a natural frequency of oscillation of the bridge structure. A well-publicized example was the Millennium Bridge over the river Thames which turned out to have sideways modes of oscillation that were excited by the swaying motion of walking. The movement of the bridge was such as to cause the pedestrians to sway in step creating a positive feedback effect.

Not all bridge structures have noticeable resonant frequencies because their motion is heavily damped: the energy is redistributed before it can build up in one mode of oscillation. If this is not the case a pronounced resonance can end up with the bridge collapsing.

Less obvious examples of resonance can be found in planetary systems. Rather than perturbing each other's motion and randomizing it, which is the more usual situation, the periods of the three inner Galilean Moons of Jupiter (Io, Europa, and Ganymede) are in simple ratios (1:2:4). Their gravitational influence on each other then holds them in stable orbits.

Two metronomes on a common platform provide a demonstration of the way in which energy is shared between resonant modes of oscillation. The metronomes are initially out of step, so the net force on the supporting structure is zero. But any small component that is in step gets enhanced and amplified until the two metronomes get locked in synchrony. In other words, energy is gradually fed into the platform at the resonant frequency of the whole system that gradually feeds back to the metronomes to bring them into resonance. We shall treat this situation more fully in Section 10.

Example: A swing, natural period T, is launched with an angular speed $-\omega$ ($\omega > 0$, $\omega \neq \frac{2\pi}{T}$) at an angle θ_0 ($\theta_0 > 0$). How high does it rise on the return? If the swing is in addition subject to a force $f\cos\omega t$, with ω close to ω_0, what is its amplitude?

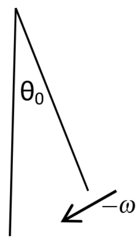


Figure 8.1: In the example, the angle of the swing is measured anticlockwise from the vertical

Assuming that the swing is an SHO the equation of motion for the angle θ with the vertical is

$$\ddot{\theta} + \omega_0^2\theta = 0,$$

where $\omega_0 = 2\pi/T$. The solution is

$$\theta = A\cos\left(\frac{2\pi}{T}t + \phi\right),$$

where A and ϕ are constants (Chapter 5).

At $t = 0$, we are given

$$\theta_0 = A\cos\phi, \tag{8.3}$$

and

$$-\omega = \left[\frac{d\theta}{dt}\right]_{t=0} = -\frac{2\pi A}{T}\sin\phi. \tag{8.4}$$

Thus,

$$\tan\phi = \frac{\omega T}{2\pi\theta_0}, \tag{8.5}$$

and

$$A = \sqrt{\frac{\omega^2 T^2}{4\pi^2} + \theta_0^2}. \tag{8.6}$$

The maximum height on the return is $\theta = A$, obtained when $\frac{2\pi}{T}t + \phi = 2\pi$, (at which point $\dot{\theta} = 0$). Thus the maximum angle is obtained from (8.6) and, from (8.5), occurs first at time

$$t = T - \frac{T}{2\pi}\tan^{-1}\frac{\omega T}{2\pi\theta_0}.$$

We should check that this is reasonable: we expect the maximum angle to be reached somewhat before a full period, which is what we have.

If the swing is also subject to a force $f\cos\omega t$, then

$$\theta = A\cos(\omega_0 t + \phi) + \frac{f}{\omega_0^2 - \omega^2}\cos\omega t \tag{8.7}$$

as can be verified by back-substitution in (8.1). (For readers familiar with the solution of differential equations, this is the sum of the complementary function and a particular integral.) At $t = 0$, we have, from (8.7) and its derivative with respect to time,

$$\theta_0 = A\cos\phi + \frac{f}{\omega_0^2 - \omega^2}, \tag{8.8}$$

and

$$-\omega = -\omega_0 A\sin\phi. \tag{8.9}$$

Comparing (8.8) with (8.3) and (8.9) with (8.4), we see that the solutions (8.5) and (8.6) are still valid provided we replace by $\theta_0 - f/\left(\omega_0^2 - \omega^2\right)$. The maximum height is obtained when $\dot{\theta} = 0$, or when t satisfies

$$A\sin\left(\omega_0 t + \phi\right) + \frac{f}{\omega_0^2 - \omega^2}\sin\omega t = 0. \tag{8.10}$$

We cannot solve this analytically in general for t, but if ω is sufficiently close to ω_0 the forcing term will dominate over the initial setup. Thus, to a first approximation, we ignore the first term in Equation (8.10) and obtain

$$t = t_1 = \frac{2\pi}{\omega},$$

for the first approximation, t_1, to the time at which the amplitude is a maximum. Rather than trying to get a better approximation for the time at maximum, it is more instructive to see that we do not need this in order to get the correction to the amplitude. Putting $t = t_1 + \delta/\omega = 2\pi/\omega + \delta/\omega$ in (8.7), expanding the cosine terms in the small quantity δ/ω, and remembering that A, like δ, has been assumed to be a quantity of first-order smallness, we have

$$\begin{aligned}\theta_{\max} &= A\cos(2\pi\left(\frac{\omega_0}{\omega}\right) + \phi + \frac{\omega_0}{\omega}\delta) + \frac{f}{\omega_0^2 - \omega^2}\cos\left(2\pi + 2\pi\delta\right) \\ &\sim A\cos\left(2\pi\left(\frac{\omega_0}{\omega}\right) + \phi\right) + \frac{f}{\omega_0^2 - \omega^2},\end{aligned}$$

and δ does not appear. The reason is that the value of a function near its maximum (or minimum) is very insensitive to the value of its argument by virtue of the fact that it is evaluated at a stationary point. This is the converse of the situation we met in Section 2.17.

For completeness, we could get a better approximation to the time at maximum displacement by putting $t = t_1 + \delta/\omega$ in (8.10) with δ small, expanding the sin functions and retaining only first-order terms. With some algebra, this gives

$$t = t_1 + \delta = \frac{A\left(\omega_0^2 - \omega^2\right)}{f}\sin\left(2\pi\left(\frac{\omega_0}{\omega}\right) + \phi\right).$$

8.2 DAMPING

Clearly Equation (8.2) cannot hold if the driving frequency ω is arbitrarily close to the natural frequency ω_0 because the amplitude cannot in practice become arbitrarily large. Either the simple harmonic approximation breaks down for large amplitudes, or we cannot neglect the dissipation of energy in some form. Here we explore this latter alternative. Rather than try to formulate a realistic mechanical model of dissipation, we construct a simple mathematical model.

A suitable model might be the exponential decay of the displacement. In that case, we should have

$$x = A\mathrm{e}^{-rt}.$$

Now we want an equation for our oscillator that involves the unknown displacement x, not an explicit function of time, so consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -rA\mathrm{e}^{-rt} = -rx,$$

and

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -r\frac{\mathrm{d}x}{\mathrm{d}t}.$$

For pure exponential decay, the acceleration depends on the velocity, $\mathrm{d}x/\mathrm{d}t$. This suggests that to incorporate damping, we modify the harmonic oscillator equation by adding a term linear in the velocity. The equation of motion of the *driven damped harmonic oscillator* becomes

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + r\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = f\cos\omega t. \tag{8.11}$$

We have not shown that any realistic physical system obeys this equation, but, conveniently, linear damping of this form is tractable mathematically and exponential decay in the absence of driving forces is a reasonable model. Note how, if we reverse the sign of the time, $t \to -t$, the equation for the SHO (8.1), without the damping term, is unchanged: we say that it is *time symmetric*. But the damping term in Equation (8.11) changes sign, so the solution of the modified equation will be quite different. Damping, as the name implies, has a direction in time.

Consider first the case with no driving force, $f = 0$.

$$\frac{d^2x}{dt^2} + r\frac{dx}{dt} + \omega_0^2 x = 0. \tag{8.12}$$

We shall go straight to the solution of the equation of motion to get the time dependence of the displacement for an oscillator undergoing damped harmonic motion. You can check the solution:

$$x = x_0\, e^{-\frac{rt}{2}} \cos(pt + \phi), \tag{8.13}$$

where

$$p = \left(\omega_0^2 - \left(\frac{r}{2}\right)^2\right)^{\frac{1}{2}} \tag{8.14}$$

by substitution of (8.13) back into Equation (8.12). Here p is the angular frequency of oscillation (in radians per unit time). Notice that the coupling to the external resistive medium alters the natural frequency and period: the new frequency p is not exactly the natural frequency ω_0, although, if the damping is not too great, they will be close.

The quantity ϕ is an arbitrary phase. This is fixed by how the oscillator is started off and not by the equation of motion. For example, a pendulum might be struck at its lowest point or dropped from an extremity. If we start the oscillator at $t = 0$ from $x = 0$, then we should have to put $\phi = \pi/2$ (since $\cos \pi/2 = 0$). The amplitude, x_0 in (8.12), is also determined by the starting conditions, for example, how hard the pendulum is struck.

Notice that the exact exponential decay of the amplitude x to $1/e$ of its original value takes place over a timescale of $2/r$, not $1/r$, as one might have expected from the approximate way in which we set up the model. Because the solution eventually decays to zero, it represents a transient phenomenon – as one might expect for a dissipative process with no driving force.

Example: A simple pendulum of mass m, length l, is set in motion from a small displacement $\theta = \theta_0$ with a linear speed v such that the small angle approximation holds at all times. If the damping force is $-mr\mathrm{d}\theta/\mathrm{d}t$. What is the condition for the pendulum to come to rest after <1 cycle?

We expect the motion to be damped in less than a cycle is the damping time $1/r$ is less than the period, $2\pi\sqrt{(l/g)}$.

The equation of motion for the angle of swing is the damped harmonic equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + r\frac{\mathrm{d}\theta}{\mathrm{d}t} + \omega_0^2\theta = 0,$$

with $\omega_0 = \sqrt{g/l}$. The general solution is

$$\theta = x_0 \mathrm{e}^{-rt}\cos(pt + \phi),$$

where

$$p = \left(\omega_0^2 - \left(\frac{r}{2}\right)^2\right)^{\frac{1}{2}}. \tag{8.15}$$

Initially, at $t = 0$, we have $\theta = \theta_0$, so

$$\theta_0 = x_0 \cos\phi.$$

Also initially, $\mathrm{d}\theta/\mathrm{d}t = v/l$ so

$$-x_0 r\cos\phi - x_0 p\sin\phi = -r\theta_0 - \left(x_0^2 - \theta_0^2\right)^{\frac{1}{2}} p = \frac{v}{l},$$

from which we can find x_0. The solution is

$$\theta = \left[\theta_0^2 + \left(\frac{r\theta_0}{p} - \frac{v}{lp}\right)^2\right]^{\frac{1}{2}} \mathrm{e}^{-rt}\cos(pt + \phi),$$

with p given by (8.14). Looking at the exponential factor e^{-pt}, the pendulum will damp in less than a cycle ($rt \sim 1$ for $t < 2\pi/p$) if

$$r > \frac{p}{2\pi},$$

or, substituting for p from (8.15) and solving for r,

$$r > \sqrt{\frac{4}{5}}\omega_0.$$

This says that the damping rate exceeds the oscillator frequency, or the damping timescale is less than a period, as we expected. (Of course, since the decay is exponential, mathematically, the pendulum never comes to rest, but the amplitude of oscillation will be small compared to the initial displacement after the specified time, here half a cycle.) We refer to this case as *critical damping* (if the decay timescale is exactly a period) or *overdamping*, if it is less.

8.3 QUALITY FACTOR

The amplitude of a damped harmonic oscillator, with damping coefficient r, falls by a factor $1/e$ in a time $2/r$. What about the energy? The energy of an SHM is

$$\begin{aligned} E &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 \\ &= \frac{1}{2}mx_0^2\omega_0^2 \sin^2(\omega_0 t + \phi) + \frac{1}{2}m\omega_0^2 x_0^2 \cos^2(\omega_0 t + \phi) \\ &= \frac{1}{2}mx_0^2\omega_0^2. \end{aligned}$$

So the energy is proportional to the square of the amplitude. For a damped oscillator, the expression is more complicated, but, if the damping is small, the differences are inconsequential, and in any case, the energy is still proportional to the square of the amplitude.

Thus, the time for the energy of an oscillator to decay by a factor of $1/e$ is $1/r$.

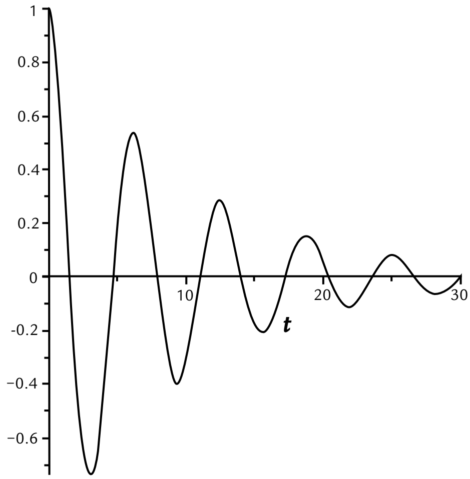


Figure 8.2: Damped harmonic motion with a moderate Q-factor: the amplitude decays appreciably in about three periods.

It is sometimes useful to express the decay time in terms of the period of oscillation: this is the *Q-value* or *quality factor* of the oscillator, which measures the time (in radians) for the energy to decay by a factor $1/e$:

$$Q = \frac{\omega_0}{r}.$$

For a bell, we want a high Q (low dissipation), for the *Tour Sans Fins*, we would want a low Q (heavy damping).

8.4 FORCED OSCILLATIONS

Let us return to Equation (8.11) including the forcing term:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + r\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = f\cos\omega t. \tag{8.16}$$

The solution consists of two contributions: one that corresponds to the free motion of the unforced oscillator and one that arises in response to the forcing term. Because the equation is linear in x, we add these solutions to obtain the general solution. However, we saw in the example in Section 8.2 that, without the forcing term, the displacement dies out in a time $1/r$. We refer to this part of the solution as transients. The transients will be the same as for the unforced oscillator and will die out in a time $1/r$. Thus, on longer timescales, we are justified in neglecting the terms in the solution that die out and in retaining only those that come from the forcing term. The time taken for the transients to die out is obviously just the time for the system to adjust to a steady state. So $1/r$ is also the timescale for the oscillator to build up the energy it absorbs from the forcing term to its steady value.

Neglecting transients, the solution of (8.16) is

$$x = f\frac{\sin(\omega t - \phi)}{\left[\left(\omega^2 - \omega_0^2\right)^2 + \omega^2 r^2\right]^{\frac{1}{2}}}, \tag{8.17}$$

where

$$\tan\phi = \frac{\omega^2 - \omega_0^2}{r\omega}. \tag{8.18}$$

This can be verified by back-substitution of (8.17) in (8.16) using the identity

$$\sin(\omega t - \phi) = \sin\omega t\cos\phi - \cos\omega t\sin\phi.$$

We shall look at how to derive this solution in Section 8.8. Note that the solution (8.17) does not depend on the initial conditions, that is, on how we set the oscillator in motion at time $t = 0$. This memory has been lost with the decay of the transients.

There are two factors in (8.17) which we shall look at separately. There is a factor multiplying amplitude f of the forcing term. There is also a phase lag $(\phi + \pi/2)$ between the driving term and the response. We shall look at each of these. (The fact that the phase lag is $(\phi + \pi/2)$ is obtained by writing $\sin(\omega t - \phi) = \cos(\omega t - \phi - \pi/2)$ and comparing this with the driving term, $\cos\omega t$; the difference is $-(\phi + \pi/2)$.)

8.5 IMPEDANCE

We rewrite the solution for our harmonically driven oscillator as

$$x = \frac{f}{Z}\sin(\omega t - \phi),$$

where the impedance Z is defined as

$$Z = \left[\left(\omega^2 - \omega_0^2\right)^2 + \omega^2 r^2\right]^{\frac{1}{2}}. \tag{8.19}$$

This has a minimum, and therefore the amplitude has a maximum where

$$0 = \frac{dZ^2}{d\omega^2} = 2(\omega^2 - \omega_0^2) + r^2.$$

(Note that to find the maximum, it is marginally easier to differentiate Z^2 rather than Z and much easier to differentiate with respect to ω^2 than ω.) Thus, resonance occurs for

$$\omega = \left(\omega_0^2 + \frac{r^2}{2}\right)^{\frac{1}{2}} \approx \omega_0 + \frac{r^2}{4\omega_0},$$

where the final expression is valid for $r << \omega_0$, that is, for large Q. This is the normal situation: otherwise, the system is heavily damped and the notion of resonance is somewhat meaningless. The resonant frequency is shifted slightly away from the natural frequency ω_0 by the coupling to a resistive medium. In addition, the amplitude is now finite at resonance.

The impedance defines the magnitude of the response of the oscillator in terms of the magnitude of the forcing term. In a simple purely resistive electrical circuit, the force would be the applied (alternating) voltage and the response would be the current. In this case, the impedance would just be the resistance of the circuit.

Note that the speed $\dot{x}$ is given by

$$\dot{x} = \frac{\omega f}{Z}\cos(\omega t - \phi).$$

The maximum amplitude is now given by minimizing $(Z/\omega)^2$, which occurs for $\omega = \omega_0$. We call this *velocity resonance*. In the case of velocity resonance, the driving frequency is equal to the natural frequency.

8.6 ENERGY AND PHASE

We have seen that the driving force and the response are in general not in phase. To discuss further, the role of phase we need first to consider the energy of the oscillator.

Since the power delivered to the oscillator is the rate of doing work we want to calculate the average value of $fv = f\dot{x}$ over a period $T = \frac{2\pi}{\omega}$. The integral

$$P = \frac{1}{T}\int_0^T \dot{x}f\,\mathrm{d}t$$

$$= \frac{f^2\omega}{Z}\int_0^T \frac{1}{T}\cos(\omega t - \phi)\cos\omega t\,\mathrm{d}t \qquad (8.20)$$

gives us the power per unit mass (since f is the force per unit mass). Before we evaluate the integral, let us see if we can get some idea of its important features from a picture. We want to know the result of integrating the product of two trigonometric functions. If these are $\pi/2$ (or $3\pi/2$) out of phase, then over a period, the product is as often negative as it is positive, so the average is zero. This occurs if the displacement and force are exactly in phase (because then the velocity and force are $\pi/2$ out of phase). If $r = 0$, then, from (8.18), $\phi = \pm\pi/2$ (depending on whether ω^2 is less than or greater than ω_0^2) so the phase difference between the displacement and forcing term is $\phi + \pi/2 = 0$ or π and we get zero dissipation, which is what we should expect.

Only if the phase difference between the force and the speed is different from $\pi/2$ will there be dissipation. At the other extreme, if the speed and force are in phase, corresponding to $\phi = 0$, there is a $\pi/2$ phase difference between the oscillator amplitude and the driving force; then the power dissipated at resonance is $f_0^2 m/2r$. In other words, dissipation is effected by introducing a phase lag between the oscillator displacement and the driving force.

The mean power supplied over a period T can be expressed in terms of the impedance by performing the integration in (8.20). We get for the power per unit mass

$$P = \frac{f^2\omega}{Z}\int_0^T \frac{1}{T}(\cos\omega t\cos\phi + \sin\omega t\sin\phi)\cos\omega t\ \mathrm{d}t.$$

$$= \frac{f^2\omega}{2Z}\cos\phi. \tag{8.21}$$

Note that in a steady state, the power supplied must equal the power dissipated.

8.7 POWER CURVE

In Figure 8.3, we show the shape of the power delivered to the oscillator as a function of the driving frequency ω from Equations (6.2) and (5.1) for $Q = 2.2$ and $Q = 10$.

Let us look at its properties. The curve peaks at the resonant frequency ω_0 (i.e., $x = 1$). As a measure of the width of the peak, we use the width of the curve at half its height. To obtain this, we proceed as follows.

Up to a constant factor, the curve is given by

$$P \propto \frac{\omega}{Z}\cos\phi = \frac{r^2\omega^2}{r^2\omega^2 + \left(\omega^2 - \omega_0^2\right)^2}. \tag{8.22}$$

At the resonant frequency, $\omega = \omega_0$, the right-hand side of (8.22) is just 1. So the frequency ω at half, the height is given by

$$\frac{r^2\omega^2}{r^2\omega^2 + \left(\omega^2 - \omega_0^2\right)^2} = \frac{1}{2},$$

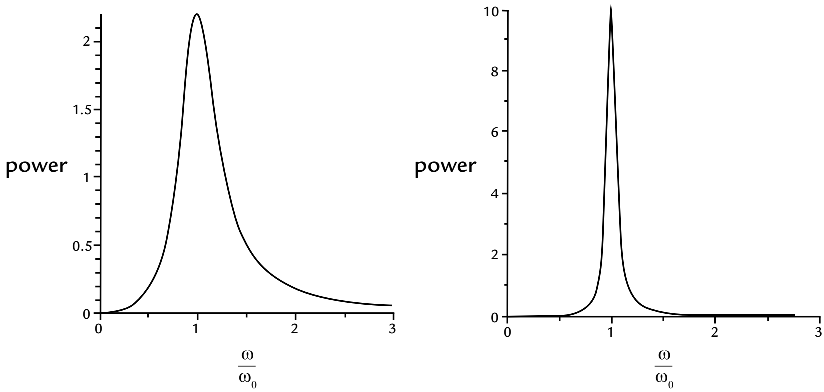


Figure 8.3: Power curve for Q = 2.2 (left) and Q = 10 (right) plotted against ω/ω_0

or

$$\omega^2 - \omega_0^2 = \pm r\omega. \tag{8.23}$$

Near $\omega = \omega_0$, we have

$$\omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0) \sim 2\omega_0(\omega - \omega_0),$$

and so we can rewrite (8.23) as

$$2\omega_0(\omega - \omega_0) \sim \pm r\omega_0.$$

It follows that $(\omega - \omega_0) = \Delta\omega = \pm\frac{r}{2}$ and the width of the peak at half height is r. The curve shows that to drive the oscillator to a significant amplitude requires a force with frequency within $r/2$ of ω_0. (You can derive this result by solving the quadratic Equation (7.2) exactly for ω and approximating the result, but there is little point in going to the trouble of obtaining an exact result only to then approximate it.)

Incidentally, we use the power curve to define the width because it is essentially symmetrical about the resonant frequency. (It cannot be exactly symmetrical because it goes to zero between $\omega = \omega_0$ and $\omega = 0$ on one side and between ω_0 and infinity on the other. We could have plotted the amplitude or velocity, in which case the shape of the curve and the widths would have been somewhat different.

Finally, we get the so-called *bandwidth theorem* by multiplying the resonance width by the decay (or rise) time:

$$\Delta\omega \times t_{\mathrm{d}} = r \times \frac{1}{r} = 1.$$

This tells us that a lightly dissipative system (small $\Delta\omega$) takes a long time to build up energy to the steady state. This is reasonable because a lightly damped system can build up a large amplitude at resonance.

Example: It is claimed that an opera singer can break a wine glass by singing a loud sustained note at the resonant frequency of the glass. Is this a reasonable claim?

According to the Metropolitan Opera Guild an opera singer can sing with a sound intensity up to I_p= 125 dB, so let us take 100 dB as a ball-park figure at the glass. This is related to air pressure by

$$p = 2 \times 10^{-5} \times 10^{\frac{I_p}{20}} = 2\,\mathrm{Nm}^{-2}.$$

This gives us the force per unit mass, pA/m, on a wineglass of area A, mass m. The power per unit mass is of order

$$P \sim \frac{f^2\omega}{2Z} \sim \frac{p^2A^2}{2rm^2}$$

since $Z = \omega r$ at resonance (Equation (5.1)). We can estimate r from the damping time: if, say τ = 5 s, then $r = 1/\tau = 0.2\ \mathrm{s}^{-1}$. The mass, excluding the base and stem (which do not resonate), is around 40 g (say). This gives a power, mP, of around 0.25 W. The stored energy is $mP\tau \sim 1.25$ J. We have to see if this is enough to break the glass. It is equivalent to dropping the glass on to a hard surface from a height $P\tau/g \sim 3$ m which is easily sufficient to break a glass.

8.8 COMPLEX EXPONENTIALS

This section is a digression on the use of complex numbers which turns out to simplify greatly the solution of differential equations like that of the damped driven oscillator (linear equations with constant coefficients). The technique of using complex numbers comes down to the fact that $e^{i\theta} = \cos\theta + i\sin\theta$ and that it is easier to manipulate exponentials than the trigonometric functions themselves, which can always be recovered by taking real or imaginary parts.

To see how this works, consider the solution of the differential equation for the driven damped oscillator but now in the form:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + r\frac{\mathrm{d}z}{\mathrm{d}t} + \omega_0^2 z = f\mathrm{e}^{i\omega t}, \tag{8.24}$$

where $x = \Re(z)$ (the real part of z) and $f\cos\omega t = \Re(f\,\mathrm{e}^{i\omega t})$. We solve the complex equation (8.24) and recover the physical solution by taking the real part at the end of the calculation.

To solve (8.24), we take a trial solution

$$z = A\mathrm{e}^{i\omega t},$$

and substitute into (8.24). This gives

$$A = \frac{f}{-\omega^2 + ir\omega + \omega_0^2}.$$

We can separate A into real and imaginary parts by multiplying denominator and numerator by $-\omega^2 - ir\omega + \omega_0^2$ (the complex conjugate of the denominator). This gives

$$A = \frac{f\left(\omega_0^2 - \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2 + r^2\omega^2} - i\frac{fr\omega}{\left(\omega_0^2 - \omega^2\right)^2 + r^2\omega^2}.$$

The real part of $A\mathrm{e}^{i\omega t}$ is

$$\frac{f\left(\omega_0^2 - \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2 + r^2\omega^2}\cos\omega t + \frac{fr\omega}{\left(\omega_0^2 - \omega^2\right)^2 + r^2\omega^2}\sin\omega t, \tag{8.25}$$

from which we can deduce the solution given in Equation (8.17) as follows. We rewrite (8.25) as

$$f\left[\left(\omega_0^2 - \omega^2\right)^2 + r^2\omega^2\right]^{-\frac{1}{2}}\left\{-\frac{\left(\omega^2 - \omega_0^2\right)}{\left[\left(\omega_0^2 - \omega^2\right)^2 + r^2\omega^2\right]^{\frac{1}{2}}}\cos\omega t + \frac{r\omega}{\left[\left(\omega_0^2 - \omega^2\right)^2 + r^2\omega^2\right]^{\frac{1}{2}}}\sin\omega t\right\}$$

$$= f\left[\left(\omega_0^2 - \omega^2\right)^2 + r^2\omega^2\right]^{-\frac{1}{2}}\{-\sin\phi\cos\omega t + \cos\phi\sin\omega t\}$$

$$= \frac{f\sin(\omega t - \phi)}{\left[\left(\omega_0^2 - \omega^2\right)^2 + r^2\omega^2\right]^{\frac{1}{2}}}$$

with

$$\tan\phi = \frac{\omega^2 - \omega_0^2}{r\omega}.$$

8.9 FOURIER ANALYSIS

Why does a tower sway in the wind and not just bend? The answer has something to do with the nonsteady nature of the wind. Even if the wind is steady upstream, the building breaks up the flow. The building is therefore subject to a time-varying force. Does the building sway in time to the force?

We have seen that an oscillator will respond with a large amplitude motion to a periodic driving force only if we impose a force near its resonant frequency. The wind, of course, does not know the resonant frequency of the building, so why does it affect the building? The reason is that any general time-varying quantity can be broken down into a sum of harmonic (sine and cosine) oscillations of different frequencies. The wind exerts a force at all frequencies but with differing amplitudes.

Let us look at a particular example. Suppose for the sake of argument that the force of the wind is periodic with a square wave form as shown in Figure 8.4. This is not realistic, but it illustrates the point. The other curves in this figure illustrate how the square wave form can be reconstructed by adding together a set of sine waves with appropriate amplitudes. For a continuous periodic signal, any desired accuracy can be achieved by taking the series to enough terms. (This is not quite true for a discontinuous function such as the square wave shown: with a finite number of terms in the series, there will always be small discrepancies between the function and the series near the points of discontinuity.)

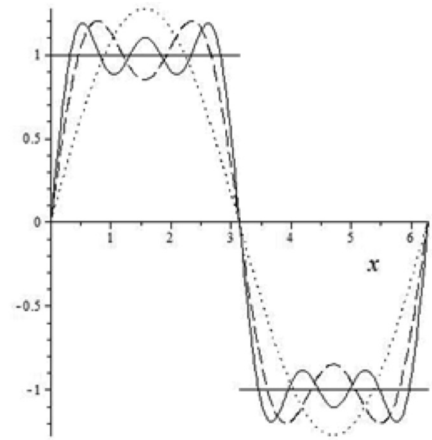


Figure 8.4: Successive approximations to a periodic square wave $x = 1$, $0 < x < \pi$; $x = -1$, $\pi < x < 2\pi$. Dotted line $y = \frac{4}{\pi}\sin x$; dashed line $y = \frac{4}{\pi}(\sin x + \frac{1}{3}\sin 3x)$; solid line $y = \frac{4}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x)$

The square wave is just one example: any periodic waveform can be approximated as a sum of harmonic functions, a result that is known as *Fourier's theorem* (and the series are called *Fourier series*). The theorem actually does more than this: it tells us how to calculate how much of each sine and cosine term to add, but we are not going to need this. In fact, Fourier's theorem also applies to nonperiodic waveforms, if we allow integrals over a continuous range of frequencies as well as sums over a discrete set.

The nonsteady wind is made up of harmonic oscillations of various frequencies and amplitudes. We see now that the component of the wind at (or near) the natural frequency of the tower causes it to sway.

8.10 COUPLED OSCILLATORS

Let us return to the *Tour Sans Fins*. To control the swaying of the building in the wind, the architect proposed the following solution: a damped pendulum suspended from the top and running down a shaft in the middle of the building that would dissipate all the wind energy absorbed by the building. How would it work? To find out we clearly need to extend our model oscillator to include two coupled oscillators, representing the building and the pendulum.

We can introduce a coupling between two oscillators by letting the force on one depend on the coordinates of the other in any way we please. A nicer model is to start from an energy equation. The case we shall deal with here is equivalent to using Hooke's law for elastic bodies. The energy for this model is given by

$$E = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\omega^2(x_1 - x_2)^2. \tag{8.26}$$

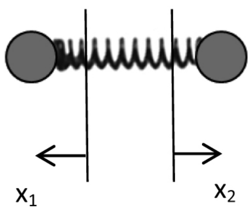


Figure 8.5: The normal mode of oscillation of two equal masses connected by a spring. The quantities x_1 and x_2 are the displacements from equilibrium of the masses. We can think of the system as two oscillators joined together by joining two springs (with their accompanying masses). In the coupled system, the masses oscillate with a common frequency with equal and opposite displacements.

Differentiating with respect to t gives us Equation (8.27):

$$\begin{aligned} 0 &= m\dot{x}_1\ddot{x}_1 + m_2\dot{x}_2\ddot{x}_2 + m\omega^2\dot{x}_1(x_1 - x_2) \\ &\quad - m\omega^2\dot{x}_2(x_1 - x_2). \end{aligned} \tag{8.27}$$

The equations of motion for the two oscillators then follow if we collect terms in $\dot{x}_1$ and $\dot{x}_2$. This gives us

$$\ddot{x}_1 + \omega^2 x_1 = \omega^2 x_2, \tag{8.28}$$

$$\ddot{x}_2 + \omega^2 x_2 = \omega^2 x_1. \tag{8.29}$$

These equations couple the two oscillators. To solve them, we need to uncouple the equations. We can do this by adding and subtracting.

Adding (8.28) and (8.29) gives us

$$\ddot{x}_1 + \ddot{x}_2 = 0,$$

or

$$x_1 = -x_2,$$

from which we see that the oscillators must move in opposite directions. This is just what we would expect from conservation of momentum: the total momentum is fixed at zero.

Subtracting (8.29) from (8.28) gives us

$$\ddot{x}_1 - \ddot{x}_2 = -2\omega^2 (x_1 - x_2),$$

which shows that the separation of the oscillators, $x_1 - x_2$, follows SHM with frequency $\sqrt{2}\omega$. This is the so-called *normal mode* of the system in which (by definition) each oscillator has the same frequency. We might call this a breathing mode.

We now vary the scenario so that the coupling strength is an independent parameter. The energy for this model is

$$E = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\omega^2 x_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\omega^2 x_2^2 - gx_1x_2,$$

where g is the coupling strength. The equations of motion are

$$\ddot{x}_1 + \omega^2 x_1 = gx_2,$$

$$\ddot{x}_2 + \omega^2 x_2 = gx_1.$$

Adding and subtracting, we get

$$(\ddot{x}_1 + \ddot{x}_2) = -\left(\omega^2 - g\right)(x_1 + x_2), \tag{8.30}$$

and

$$(\ddot{x}_1 - \ddot{x}_2) = -\left(\omega^2 + g\right)(x_1 - x_2). \tag{8.31}$$

The normal modes have frequencies $\omega_+ = \left(\omega^2 - g\right)^{\frac{1}{2}}$ and $\omega_- = \left(\omega^2 + g\right)^{\frac{1}{2}}$. The effect of the coupling is to perturb the natural frequencies of the oscillators. To see the behavior in more detail, we solve (8.30) and (8.31) to get

$$x_1 = a\cos\left(\omega_+ t + \phi_+\right) + b\cos\left(\omega_- t + \phi_-\right),$$

$$x_2 = a\cos\left(\omega_+ t + \phi_+\right) - b\cos\left(\omega_- t + \phi_-\right). \tag{8.32}$$

To see what is happening, we need a small mathematical trick. We write the solution for x_1 as

$$x_1 = \frac{a+b}{2}\cos\left(\omega_+ t + \phi_+\right) + \frac{a+b}{2}\cos\left(\omega_- t + \phi_-\right) + \frac{a-b}{2}\cos\left(\omega_+ t + \phi_+\right) - \frac{a-b}{2}\cos\left(\omega_- t + \phi_-\right),$$

with a similar expression for x_2 with the replacement $b \to -b$. Then, using the trigonometric identities

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2},$$

and assuming $\omega^2 \gg g$, we get

$$x_1 = (a+b)\cos(\omega t + \phi_+)\cos\left(\frac{gt}{2\omega} + \phi_-\right) - (a-b)\sin(\omega t + \phi_+)\sin\left(\frac{gt}{2\omega} + \phi_-\right)$$

$$x_2 = (a-b)\cos(\omega t + \phi_+)\cos\left(\frac{gt}{2\omega} + \phi_-\right) - (a+b)\sin(\omega t + \phi_+)\sin\left(\frac{gt}{2\omega} + \phi_-\right), \tag{8.33}$$

where we have approximated

$$\omega_+ \sim \omega - \frac{g}{2\omega},$$

and

$$\omega_- \sim \omega + g/2\omega,$$

and we have set

$$\phi_+ = \frac{(\phi_+ + \phi_-)}{2}, \; \phi_- = \frac{\phi_+ - \phi_-}{2}.$$

Note that x_2 is obtained from x_1 by setting $b \to -b$ in accordance with Equations (8.32). Each factor in Equations (8.33) is a product of a rapid oscillation (frequency ω) and a slower oscillation (frequency $g/2\omega$). This enables us to picture the motion. The masses oscillate at the resonant frequency ω, with an envelope or amplitude that has a period $4\pi\omega/g$. The energy is therefore exchanged between the oscillators on this timescale, $2\pi\omega/g$, which is much greater than the natural period $2\pi/\omega$.

If we set one oscillator in motion, while the other remains at rest, we see that energy will be transferred back and forth between the oscillators. This is the situation we described in Section 8.1.

Note that if $a \neq 0$, we no longer have $\dot{x}_1 = -\dot{x}_2$, so momentum is apparently not conserved. There are two ways of looking at this. The first is that the forces in the system are not independent of position (because the equations of motion refer to coordinate values and not just differences); in this case, we would not expect momentum to be conserved. The other way of looking at it is to enlarge the system to include the spring: then the overall conservation of momentum is preserved by a flow of momentum along the spring. This is an important general point: whenever action at a distance is involved a full analysis requires all components of the system to be considered.

8.11 COUPLED OSCILLATORS WITH DISSIPATION

We now add a dissipative term to one of the oscillators, so the equations of motion become

$$\ddot{x}_1 + r\dot{x}_1 + \omega_1^2 x_1 = g x_2, \tag{8.34}$$

$$\ddot{x}_2 + \omega_2^2 x_2 = g x_1. \tag{8.35}$$

We look for a solution of the form

$$x_1 = a_1 e^{i\omega t};\ x_2 = a_2 e^{i\omega t}. \tag{8.36}$$

Substituting (8.36) in to (8.34) and (8.35), we get

$$\left(-\omega^2 + \omega_1^2 + i\omega r\right) a_1 - g a_2 = 0$$

$$-g a_1 + \left(-\omega^2 + \omega_2^2\right) a_2 = 0. \tag{8.37}$$

For $a_1 a_2 \neq 0$, we require the determinant of the coefficients in (8.37) to vanish:

$$\left(-\omega^2 + \omega_1^2 + i\omega r\right)\left(\omega_2^2 - \omega^2\right) - g^2 = 0, \tag{8.38}$$

or

$$\omega^4 - ir\omega^3 - \left(\omega_1^2 + \omega_2^2\right)\omega^2 + i\omega_2^2 r\omega + \omega_1^2\omega_2^2 - g^2 = 0. \tag{8.39}$$

(You can verify (8.38) by elementary means by solving the pair of simultaneous equations for the unknown amplitudes a_1 and a_2: you will find that $a_1 = a_2 = 0$ unless (8.38) holds.) We now assume that the coupling and the resistance are small. Our first approximation, from (8.38), is

$$\left(\omega^2 - \omega_1^2\right)\left(\omega^2 - \omega_2^2\right) = 0.$$

So we have two solutions: either $\omega \sim \omega_1$ or $\omega \sim \omega_2$. In the first case, our next approximation is (from (8.38))

$$\omega^2 \sim \omega_+^2 = \omega_1^2 + \frac{g^2 + i\omega_1 r\left(\omega_1^2 - \omega_2^2\right)}{\omega_1^2 - \omega_2^2},$$

and from (8.37)

$$a_2 = -a_1 \frac{g}{\omega_1^2 - \omega_2^2}.$$

Similarly, if $\omega \sim \omega_2$, we obtain

$$\omega^2 \sim \omega_-^2 = \omega_2^2 + \frac{g^2}{\omega_1^2 - \omega_2^2},$$

and

$$a_2 = a_1 \frac{\omega_1^2 - \omega_2^2}{g}.$$

The general solution is then

$$x_1 = a_1 \cos\left[\omega_1\left(1 - \frac{1}{2}\frac{g^2}{\omega_1^2\left(\omega_2^2 - \omega_1^2\right)}\right)t\right]e^{-\frac{rt}{2}}$$

$$+ b\cos\left[\omega_2\left(1 + \frac{1}{2}\frac{g^2}{\omega_1^2\left(\omega_2^2 - \omega_1^2\right)}\right)t\right]$$

$$x_2 = a_1 \frac{\omega_1^2 - \omega_2^2}{g}\cos\left[\omega_1\left(1 - \frac{1}{2}\frac{g^2}{\omega_1^2\left(\omega_2^2 - \omega_1^2\right)}\right)t\right]e^{-\frac{rt}{2}}$$

$$+ \frac{bg}{\omega_2^2 - \omega_1^2}\cos\left[\omega_2\left(1 + \frac{1}{2}\frac{g^2}{\omega_1^2\left(\omega_2^2 - \omega_1^2\right)}\right)t\right]$$

Let us see what it tells us: the $\omega_+ \sim \omega_1$ mode decays on a timescale $2/r$ leaving the $\omega_- \sim \omega_2$ mode oscillating forever. This cannot be correct. The coupling should be feeding energy from x_2 to x_1 where it is dissipated. There is, in the solution, we have obtained so far, no timescale over which this happens. The problem is clearly that the ω_2 mode is not dissipative. In fact, we can see that there is no phase difference between x_1 and x_2 in this mode, so there is no transfer of energy between the oscillators. This means that we have not taken the approximation far enough. A more accurate expression for ω_- is obtained in the next approximation:

$$\omega \sim \omega_- = \omega_2 + \frac{g^2}{\omega_1^2 - \omega_2^2}\left[1 + \frac{ir}{2\omega_2\left(\omega_2^2 - \omega_1^2\right)}\right],$$

and

$$a_2 = \left[\frac{\omega_1^2 - \omega_2^2}{g} - \frac{ir}{g}\right]a_1,$$

giving a phase difference

$$\tan\phi = \frac{r}{\omega_1^2 - \omega_2^2}.$$

The ω_- mode is now dissipative, transferring energy to x_1 on a tim-escale $(\omega_2^2 - \omega_1^2)^2/rg^2$.

Note that there is an apparent singularity at $g = 0$. However, $g = 0$ breaks the conditions of the approximation. It would lead to $\omega_1 = \omega_2$, but we must have

$$\left|\frac{g^2}{\omega_1^2 - \omega_2^2}\right| \ll \omega_1^2 \text{ and} \left|\frac{g^2}{\omega_1^2 - \omega_2^2}\right| \ll \omega_2^2$$

in order that the frequencies are perturbed by small amounts. Thus, ω_1 and ω_2 must be sufficiently different, whence $g \neq 0$. If the oscillators are not coupled ($g = 0$), then the behavior is quite different.

8.12 FORCED COUPLED OSCILLATORS

Consider finally a model of the pendulum in the tower: the case where the undamped oscillator is subject to an external periodic force. We know that the system will absorb energy from the driving force only if it is near resonance, so we need only consider this case. (Only the resonant frequencies in the wind cause the building to sway.) To gain some insight, consider first a simple model of masses on a spring. Suppose the masses are very unequal and the larger mass is damped. At the larger mass most of the energy traveling along the spring will be reflected. Thus, there will be very little damping. Alternatively, consider the case where the smaller mass is damped. Since its energy is much less than that of the larger mass, however, effectively it drains energy from the larger mass its rate of dissipation is limited by the relatively small store it has to dissipate at any time. Thus, we expect that we shall have effective dissipation only if the masses are close. This is an example of *impedance matching*.

For our general oscillators, we model this case by first taking $g_1 \neq g_2$. The aim is then to show that damping is effective if, in fact, $g_1 \sim g_2$. For simplicity, we assume that $\omega_1 = \omega_2 = \omega_0$ although this is not essential. The equations of motion are therefore

$$\ddot{x}_1 + r\dot{x}_1 + \omega_0^2 x_1 = g_1 x_2 \tag{8.40}$$

$$\ddot{x}_2 + \omega_0^2 x_2 = g_2 x_1 + f e^{i\omega t} \tag{8.41}$$

where we assume the real parts of all quantities are implied. We shall also assume that the damping constant r is small ($r \ll \omega_0^2$) and that the coupling is relatively weak so that it can be treated as a perturbation to the harmonic oscillations.

We have seen in Section 8.11 that the unforced motion of the oscillators is damped. This means that we can neglect the effects of initial conditions and consider the steady state driven entirely by the forcing term. We therefore look for solutions of the form

$$x_1 = a_1 e^{i\omega t}; \quad x_2 = a_2 e^{i\omega t}.$$

If we neglect the effect of the damping on the resonant frequency, we find

$$\omega^2 = \omega_0^2 \pm \sqrt{g_1 g_2}.$$

With this value of ω, we can now solve (8.41) and (8.42), to a first approximation:

$$a_2 = f\left(1 - i\frac{\sqrt{(g_1 g_2)}}{\omega r}\right),$$

and

$$a_1 = \frac{i g_1}{\omega r}.$$

The dissipation timescales are equal if

$$\frac{\sqrt{(g_1 g_2)}}{\omega r} = \frac{g_1}{\omega r},$$

or $g_1 = g_2$. For our masses on a spring, this means that the dissipation is most effective if the masses are equal, as we proposed. This is a special example of a more general result that energy transfer is most effective if the impedances of the systems are matched. Our conclusion is that the pendulum will effectively suppress the motion of the tower in the wind if the tower and the pendulum are impedance matched.

8.13 CHAPTER SUMMARY

- The equation of motion for the displacement of a damped, driven harmonic oscillator is

$$\ddot{x} + r\dot{x} + \omega_0^2 x = f_0 e^{i\omega t},$$

  where the real parts of complex quantities are to be taken.

- Impedance is the ratio of the driving term to the response (displacement)
- Resonance occurs when an oscillator is driven by a periodic force close to its natural frequency; the resonant frequency occurs at minimum impedance.
- The quality factor Q of an oscillator is defined by $Q = \frac{\omega}{r}$ and is the number of periods for the amplitude to decline by a factor of e.
- In a damped oscillator, the driving term and displacement (or velocity) are out of phase. If the phase difference is ϕ, the rate of dissipation is proportional to $\cos\phi$.

8.14 EXERCISES

1. Once in motion the amplitude of a child's swing can be increased by pumping the oscillation by adjusting position in time with the swing. Because the effect is to alter the

parameters of the swing (its natural length or moment of inertia), the process is called parametric amplification or parametric resonance.

Consider therefore an SHO subject to a periodic variation of its natural frequency:

$$\ddot{x} + \omega_0^2(1 + \varepsilon \sin 2\omega_0 t)x = 0, \tag{8.42}$$

where $\varepsilon \ll 1$. If we neglect the perturbation the solution is, say,

$$x = A\cos(\omega t + \phi).$$

Setting $\phi = 0$ for simplicity show that (8.42) can be written

$$\ddot{x} + \omega_0^2 x = -\frac{1}{2}\varepsilon\omega_0^2(\sin\omega_0 t + \sin 3\omega_0 t).$$

Show that this gives rise to a linearly growing solution for $x(t)$. Show further that the power supplied over a cycle is given by

$$P = \frac{1}{4}\varepsilon\omega_0^3 A^2,$$

and hence that the energy of the oscillator grows exponentially.

2. A spherical buoy floats half-submerged in water and is observed undergo small oscillations about this equilibrium position with frequency ω. Show that the mass of the buoy is

$$\frac{9}{4}\pi g^3\omega^6\rho,$$

where ρ is the density of water and g is the acceleration due to gravity.

3. Two equal masses are joined by a spring. One of the masses is highly damped. The other mass is set in motion. Investigate the behavior of the system.

4. Estimate the Q-value of a tuning fork.

5. Show that velocity resonance occurs when a damped harmonic oscillator is driven at its natural frequency (Section 8.5).

6. Fill in the missing steps in Section 8.12.

7. An oscillator with displacement satisfying the equation of motion

$$\ddot{x} + r\dot{x} + \omega_0^2 x = 0$$

is critically damped if $r^2 = 4\omega_0^2$. Show that in this case the solution of the equation of motion is

$$x = e^{-\frac{rt}{2}}(at + b),$$

where a and b are arbitrary constants.

CHAPTER 9

Rigid Bodies

Problem: Galileo is credited with establishing that all bodies fall with the same acceleration under gravity. He did this not by dropping bodies from the leaning tower of Pisa, as legend has it, but by rolling balls down an inclined plane. This has the advantage of diluting gravity, which makes it easier to measure the time of fall with the methods available to Galileo. However, Galileo was fortunate in the shapes of bodies he chose to compare.

Picture credit: David Wilmot (Creative Commons)
(http://commons.wikimedia.org/wiki/File%3ALeaning_Tower_of_Pisa_(1).jpg).

Does the shape of the rolling object make any difference? Would Galileo have made his discovery if he had compared the rolling motion of spheres and cylinders?

9.1 ROTATIONAL ENERGY

When we consider the kinetic energy of a body in linear motion, we do not have to worry about the distribution of mass because all parts of the body are moving with the same speed. But the different parts of an extended body that is rotating are all moving at different speeds, so we cannot say that the kinetic energy is just $\frac{1}{2}mv^2$.

To get the correct formula, consider a point mass, δm, rotating about a point O at a distance r with speed v. Its angular speed is $\omega = v/r$. Its kinetic energy is $\frac{1}{2}\delta m v^2 = \frac{1}{2}\delta m r^2\omega^2$. Now consider a rigid body rotating about O. All parts of the body have the same ω but their distances from O will be different. Thus, the kinetic energy will be

$$E_{\mathrm{K}} = \frac{1}{2}\left(\sum \delta m r^2\right)\omega^2,$$

where the sum is over all the elements of mass in the body. We write this as

$$E_{\mathrm{K}} = \frac{1}{2}I\omega^2,$$

where I is called the moment of inertia of the body. In practice, we work out I as an integral:

$$I = \int r^2 \mathrm{d}m = \int r^2 \frac{\mathrm{d}m}{\mathrm{d}r}\mathrm{d}r.$$

9.2 MOMENTS OF INERTIA

Start with a simple example: the moment of inertia of a rod of length l pivoted at one end. We have $\mathrm{d}m = \rho \mathrm{d}x$, so

$$I = \int_0^l x^2 \rho \,\mathrm{d}x = \frac{1}{3}ml^2. \tag{9.1}$$

We might have been tempted to guess that we could replace the rod by an equal mass at its CM. This would give $I = m(l/2)^2$. Why is the factor in (9.1) 1/3 and not 1/4? The reason is that the factor of x^2 in the integral (9.1) for the moment of inertia weights the contributions of segments of the rod toward the more distant contributions. The equivalent mass is not at the CM but a bit farther out.

Next another simple example: the moment of inertia of a disc:

$$I_{\mathrm{D}} = \int_0^R r^2 2\pi r\,\rho\,\mathrm{d}r = \frac{1}{2}\pi R^2 \rho R^2 = \frac{1}{2} M_{\mathrm{D}} R^2.$$

To work out the rotational energy of the Earth, therefore, we need its moment of inertia. To get this accurately involves knowledge of the radial mass distribution $m(r)$. However, we can estimate it by taking the Earth to be a uniform sphere, of density ρ, radius R_{E}, and mass M_{E}.

Moment of inertia of a uniform sphere:

$$I_{\mathrm{S}} = \int_{-R}^{R} \frac{1}{2} M_{\mathrm{D}} r(z)^2\,\mathrm{d}z = \int_{-R}^{R} \frac{1}{2}\rho\pi\, r(z)^4\,\mathrm{d}z$$
$$= \int_{-R}^{R} \frac{1}{2}\rho\pi \left(R^2 - z^2\right)^2 \mathrm{d}z = \frac{2}{5} M_{\mathrm{E}} R_{\mathrm{E}}^2,$$

where the final integral can be obtained either by expanding the bracket or with the substitution $z = R\cos\theta$.

Moment of inertia of a uniform cylinder:

For a uniform cylinder of radius R, height H rotating about its axis of symmetry, we have

$$I_{\mathrm{C}} = \int_0^H \int_0^R r^2 2\pi r\rho\;\mathrm{d}r\;\mathrm{d}h = \frac{1}{2}\pi R^4 H\rho = \frac{1}{2} M R^2.$$

About a perpendicular axis through the center, we have

$$I_{\mathrm{CM}} = 2\int_0^{H/2} h^2 \pi R^2 \rho\,\mathrm{d}h = \frac{1}{12} H^3 \pi R^2 \rho = \frac{1}{12} M H^2.$$

And about a perpendicular axis through one end:

$$I_E = \int_0^H h^2 \pi R^2 \rho \, dh = \frac{1}{3} H^3 \pi R^2 \rho = \frac{1}{3} M H^2.$$

Note that

$$I_E = I_{CM} + M\left(\frac{H}{2}\right)^2.$$

This is an example of the *parallel axis theorem*:

$$I = I_{CM} + Mr^2$$

for the moments of inertia about the CM and about a parallel axis a distance r apart.

Example: Find the moment of inertia I_V of a square laminar of side $2a$ of surface density σ about one vertex with respect of an axis normal to the plane of the square.

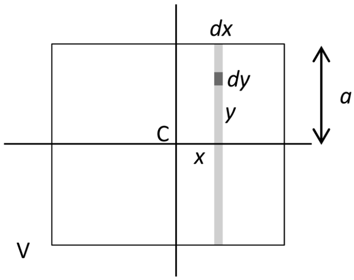


Figure 9.1: Moment of inertia of a square laminar

Consider first a strip width dx at x from the center. The moment of inertia about the center is

$$dI_x = \sigma \, dx \int_{-a}^{a} y^2 dy = \frac{2\sigma a^3}{3} dx.$$

Using the parallel axes theorem, the moment of inertia of this strip about the center is

$$dI_C = dI_x + (2a\sigma \, dx) x^2.$$

The moment of inertia of the square about the center is

$$I_C = \int_{-a}^{a} dI_x + 2a\sigma x^2 dx = \int_{-a}^{a} \left(\frac{2\sigma a^3}{3} + 2a\sigma x^2 \right) dx = \frac{8}{3}\sigma a^4.$$

Using the parallel axes theorem again, with M the mass of the laminar,

$$I_V = I_C + M\left(\sqrt{2}a\right)^2 = 4a^2\sigma\left(2a^2\right) + \frac{8}{3}\sigma a^4 = \frac{32}{3}\sigma a^4 = \frac{8}{3}Ma^2.$$

Example: Estimate the rotational energy of the Earth.

We have for the rotational energy, E_R

$$E_R = \frac{1}{2}I\omega^2 = \frac{1}{5}M_E R_E^2 \omega^2 \sim 0.2\times 6\times 10^{24}\times\left(6\times 10^6\right)^2\times\left(\frac{2\pi}{3\times 10^7}\right)^2 \sim 2\times 10^{24}\,\mathrm{J}.$$

To get some idea of the size of this number, we can compare it to the world's energy consumption. It is sufficient to provide the current world demands for power for about a billion years. It is also about a million times the energy falling on the Earth from the Sun in a year.

Of somewhat more practical use, we could estimate the energy that can be stored in a flywheel, for example, for regenerative braking in a car or bus, or for smoothing out peaks in demand on the national electricity grid. Current technology allows rotation rates in excess of 25 000 rpm and is able to provide an energy store of 400 kJ per kg using composite materials.

9.3 ANGULAR MOMENTUM

The angular momentum of a point mass δm velocity $\boldsymbol{v}$ with position vector $\boldsymbol{r}$ is $\delta m(\boldsymbol{r} \wedge \boldsymbol{v})$. For a rigid body, angular velocity $\boldsymbol{\omega}$, this becomes

$$\boldsymbol{H} = \sum \delta m\,\boldsymbol{r} \wedge \boldsymbol{v} = \sum \delta m\,\boldsymbol{r} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{r}) = \boldsymbol{\omega} \sum \delta m r^2 = I\boldsymbol{\omega}.$$

Expanding the triple vector product gives us

$$\boldsymbol{H} = \Sigma\delta m\, \boldsymbol{r}^2\, \boldsymbol{\omega} - \Sigma\delta m\, \boldsymbol{r}\, (\boldsymbol{r} \cdot \boldsymbol{\omega}).$$

For a planar body rotating about an axis perpendicular to the plane $\boldsymbol{r}$ and $\boldsymbol{\omega}$ are orthogonal, so the final term vanishes. More generally, we write $\boldsymbol{r}$ as a sum of components parallel to $\boldsymbol{\omega}$ and perpendicular to ω: $\boldsymbol{r} = \boldsymbol{r}_{\parallel} + \boldsymbol{r}_{\perp}$. Then the final term is

$$\Sigma\delta\, m\Big[\Big(r_{\parallel}^2 + r_{\perp}^2\Big) - \Big(\boldsymbol{r}_{\parallel} + \boldsymbol{r}_{\perp}\Big)\Big(r_{\parallel}\omega\Big)\Big].$$

Now, if the body is rotating about an axis of symmetry,

$$\Sigma\delta\, m \boldsymbol{r}_{\perp}\Big(r_{\parallel}\omega\Big) = 0,$$

because contributions from opposite sides of the axis ($r_{\rightarrow} \rightarrow -r_{\rightarrow}$) will cancel. Also, $\boldsymbol{r}_{\uparrow}(r_{\uparrow}\omega) = r_{\uparrow}\hat{\omega}(r_{\uparrow}\omega) = r_{\uparrow}{}^2\boldsymbol{\omega}$, so collecting terms leaves

$$H = \Sigma\delta m\, r_{\perp}{}^2\, \boldsymbol{\omega}.$$

So for a body rotating about an axis of symmetry, the angular momentum is

$$H = I\omega, \tag{9.2}$$

where $I = \Sigma\delta m\, r^2 \sin^2\theta$ with θ the angle between the axis and the position vector to δm. The quantity I is called the moment of inertia of the body about the symmetry axis. If the rotation is not about an axis of symmetry, the relation between the angular momentum and angular velocity is still given by (9.2), but the moment of inertia is then represented by a 3 × 3 matrix which can be obtained by writing the equation out in component form. We get

$$H_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z,$$

where $I_{xx} = \Sigma\delta m\, (y^2 + z^2)$, $I_{xy} = -\Sigma\delta m\, xy$, etc. with corresponding equations for H_y and H_z.

If there is no net external couple on a system, angular momentum is conserved. We can see this as follows:

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = \sum \delta m\left(\boldsymbol{v} \wedge \boldsymbol{v} + \boldsymbol{r} \wedge \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right) = \sum \delta m \boldsymbol{r} \wedge \boldsymbol{F} = M\boldsymbol{R}_{\mathrm{CM}} \wedge \boldsymbol{F}, \quad (9.3)$$

which is zero if there is no net couple ($\boldsymbol{R}_{\mathrm{CM}} \wedge \boldsymbol{F} = 0$).

9.4 THE RECEDING MOON

The Moon recedes from the Earth at the rate of 3.8 cm a year as a result of tidal torques exerted by the Earth on the Moon. How can we explain this? Estimate the resulting change in the length of the day.

Let H_{M} be the magnitude of the angular momentum of the Moon, and let the Earth have moment of inertia I_{E} and angular speed ω_{E}. Conservation of angular momentum implies

$$\delta H_{\mathrm{M}} = -I_{\mathrm{E}}\delta\omega_{\mathrm{E}}. \quad (9.4)$$

But the change in angular momentum of the Moon, if its distance changes by δR, is

$$\delta H_{\mathrm{M}} = Mv\delta R = \frac{2\pi R}{P_{\mathrm{M}}} M\ \delta R. \quad (9.5)$$

where P_{M} is the orbital period of the Moon. Equating the two expressions for δH_{M} gives

$$\delta\omega_{\mathrm{E}} = \frac{2\pi R}{P_{\mathrm{M}} I_{\mathrm{E}}} M\ \delta R. \quad (9.6)$$

If P_{E} is the length of the day, then

$$\delta\omega_{\mathrm{E}} = \delta\left(\frac{2\pi}{P_{\mathrm{E}}}\right) = -\frac{2\pi}{P_{\mathrm{E}}^2}\delta P_{\mathrm{E}} = -\frac{2\pi R}{P_{\mathrm{M}} I_{\mathrm{E}}} M\delta R,$$

where the final equality follows from (9.6). Thus, with $I_{\mathrm{E}} = \frac{2}{5} M_{\mathrm{E}} R_{\mathrm{E}}^2$,

$$\frac{\delta P_E}{P_E} = \frac{P_E}{P_M}\frac{MR\delta R}{I_E} = \frac{P_E}{P_M}\frac{5M}{2M_E}\frac{R\delta R}{R_E^2} \sim \frac{1}{28}\times\frac{5}{2}\times\frac{1}{60}\times\frac{380\,000}{6000^2}0.038 \sim 5\times10^{-7}.$$

The fractional change in the length of the day is 5×10^{-7} over a year, so the day decreases by 0.04 s a year.

9.5 SPACE TETHER

In near Earth orbit, the atmosphere is still dense enough to produce a drag force on the international space station. In order to remain in orbit, the space station must therefore be supplied with energy. An efficient way of doing this is to use the energy of visiting space shuttles as they undock, by paying out the shuttle on a long tether. How would this work?

Given that the dissipation of energy is small, we can ignore these frictional losses in considering the orbits. Therefore, the total energy must be conserved. Since there are no external couples acting on the system, the angular momentum must be conserved. This gives us a clue as to what is proposed. If the shuttle is losing angular momentum as well as energy, then that angular momentum and energy must be transferred to the space station. What affect will this have: one might guess that it will boost the space station to a higher orbit.

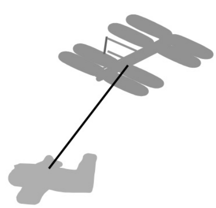


Figure 9.2: A shuttle tethered to a space station

We give the shuttle a little nudge toward the Earth to start things off. Looked at in the rotating reference frame, there is a net downward force on the lower mass, here the shuttle, since gravity closer to the Earth is greater than the centrifugal force, and an outward one on the upper mass, here the space station, since gravity is less than the centrifugal force. Since the link between them is not rigid, these forces will cause the tether to be paid out without any further expenditure of energy.

As the shuttle falls, its kinetic energy and potential energy change. In a circular orbit $E_K = -E_P/2$, so half the potential energy it loses goes into kinetic energy. An increase in kinetic energy means that the shuttle speeds up. This is correct: bodies closer in go round faster. But where has the other half of the change in potential gone? The only possibility is into increasing the energy of the space station.

9.6 EQUATION OF MOTION

Equation (9.2) gives us the equation of motion of a rotating system that is subject to external torques:

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = \boldsymbol{G},$$

or, more simply, for rotation about a fixed axis of symmetry,

$$I\frac{\mathrm{d}\omega}{\mathrm{d}t} = G,$$

or

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = G.$$

In the spirit of Chapter 5, we can derive this from the energy of the system. We have

$$E = E_K + E_U = \frac{1}{2}I\omega^2 + U(\theta),$$

and hence,

$$0 = \frac{\mathrm{d}E}{\mathrm{d}t} = I\omega\frac{\mathrm{d}\omega}{\mathrm{d}t} + \frac{\mathrm{d}U}{\mathrm{d}\theta}\omega.$$

Identifying $G = -\mathrm{d}U/\mathrm{d}\theta$ gives us the equation of motion.

9.7 COMPOUND PENDULUM

A simple pendulum is a point mass on a massless string undergoing small angle oscillations. If we make the support rigid and massive, we have a compound pendulum. How would we expect this to change the period? For simplicity, consider a rod of length l mass m pivoted at one end. The only quantities that can enter the expression for the period are again m, l, and g. So the period is proportional to $(l/g)^{\frac{1}{2}}$ but with a different constant of proportionality. To receive this, we have to solve the equations of motion.

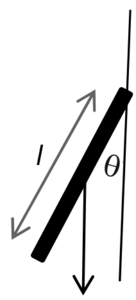


Figure 9.3: A compound pendulum

We have

$$G = mg\frac{l}{2}\sin\theta = I\ddot{\theta}$$

where $I = 1/3\, ml^2$. Thus,

$$\ddot{\theta} = \frac{3}{2}\frac{g}{l}\theta$$

for small oscillations and the period is

$$2\pi\sqrt{\frac{2l}{3g}}.$$

The effect of adding mass nearer the pivot is to shorten the period as we might expect.

9.8 A MODEL OF RUNNING

The following is an extract from the Harvard University Gazette (April 30, 1998):

> Sternlight guessed that an upper limit on the frequency of stride might restrict a person's running speed. She measured stride frequency and length, the amount of time a runner's foot is in contact with the ground, and the time each foot is in the air. The latter is called "swing time."
>
> To Sternlight's amazement, whether people ran fast or slow, or whether they ran uphill or downhill, everyone had approximately the same swing time at top speed.
>
> Those running 14 miles an hour and those running 27 miles an hour both took between 0.37 and 0.40 second to swing one leg in front of the other.
>
> "What limits top speed, then, is the minimum time you take to swing your leg into position for the next step," Sternlight concludes. "That's evidently a fundamental limit for all humans. What determines how fast you can run is how fast you're going when you reach that limit."

According to this theory, the leg is a compound pendulum. The average leg length is 0.95 m, so the swing time (one half a period) becomes 1.96/2 = 0.98 s. The typical stride length of a sprinter is 1 m at a speed of 10 m s^{-1} so a stride time of 0.1 s, very different form 0.98 s. Clearly, there is something wrong! The pendulum model is satisfactory for slow walking (0.5 m s^{-1} with a step of 0.3 m giving a step period of 0.6 s which is somewhat closer to 0.98 s) but running is clearly qualitatively different from fast walking.

9.9 ROLLING AND SLIPPING

Return now to a question we left unanswered in Chapter 2. Under what conditions does a cart wheel roll rather than slip as the horse and cart move off. What do we expect? Clearly, we must not accelerate away too quickly, so we are seeking a limit on the initial acceleration. This could in principle depend on all the parameters in the problem: the mass of the cart, M; the radius of a wheel, r; the acceleration due to gravity, g; and the coefficient of friction μ. However, dimensionally, we are restricted to $a \sim f(\mu)g$, where f is some unknown function. This is as far as we can get without a calculation.

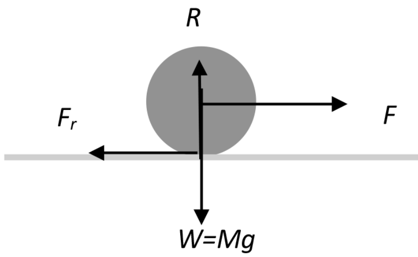


Figure 9.4: Does the wheel slip before rolling?

So consider the situation in Figure 9.4, where now the wheel is not treated as a point mass, and the points of application of the forces therefore matter.

Resolving the forces vertically, we have

$$R = Mg. \tag{9.7}$$

Resolving horizontally, we have

$$F - F_r = Ma. \tag{9.8}$$

Taking moments about the point of contact between the wheel and the ground:

$$Fr = I\dot{\omega}, \tag{9.9}$$

where I is the moment of inertia of the wheel. For there to be no slipping,

$$r\dot{\omega} = a,$$

so, from (9.9),

$$Mar + rF_r = Ia.$$

Putting $F_r = \mu Mg$ and $I = \frac{1}{2}Mr^2$ and solving for a, rolling will occur before slipping if

$$a < 2\mu g.$$

9.10 GALILEO'S INCLINED PLANE

We can now tackle the problem of Galileo's inclined plane. The task is to determine the acceleration down the plane of rolling objects of different shapes.

We can determine the equation of motion of a rolling body from the conservation of energy. Suppose the body has mass M, moment of inertia I, radius a, speed v, and is at vertical distance z below its starting point. The total energy E of the body is given by the sum of the translational kinetic energy, the rotational energy, and the potential energy:

$$E = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 - Mgz = \text{constant},$$

where $\omega = v/a$. Thus,

$$0 = \frac{\mathrm{d}E}{\mathrm{d}t} = Mv\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{I}{a^2}\,v\frac{\mathrm{d}v}{\mathrm{d}t} - Mg\frac{\mathrm{d}z}{\mathrm{d}t},$$

where if the body rolls on a slope that makes an angle α with the horizontal,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v\sin\alpha.$$

The acceleration is therefore

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{g\sin\alpha}{1 + \dfrac{I}{Ma^2}}, \tag{9.10}$$

which is constant. The constant acceleration formulae, starting from rest, give the travel time:

$$T = \left(\frac{2z}{g\sin^2\alpha}\right)^{\frac{1}{2}}\left(1 + \frac{I}{Ma^2}\right)^{\frac{1}{2}}.$$

Comparing objects of similar shape, we have I/Ma^2 = constant, so the times are the same, independent of mass and radius. But comparing spheres and cylinders, we have

$$\frac{T_s}{T_c} = \frac{\left(1+\frac{2}{5}\right)^{\frac{1}{2}}}{\left(1+\frac{1}{2}\right)^{\frac{1}{2}}} = \left(\frac{14}{15}\right)^{\frac{1}{2}},$$

a 3% difference, probably beyond the accuracy that Galileo could achieve at a time when there were no accurate clocks.

Suppose we had not thought of starting from energy conservation. How would we apply Newton's law's directly?

Taking moments about the point of contact, we have

$$G = Mga \sin\alpha = I_0\dot{\omega},$$

where $I_0 = I + Ma^2$ is the moment of inertia about the point of contact and $v = a\omega$. Dividing through by Ma^2 gives (9.10).

9.11 SPIN AND PRECESSION

Consider a spinning top or gyroscope. It not only spins about its axis, but this axis of rotation will in general rotate about some fixed direction. The rotation of the axis is called precession. Why then does a spinning top precess?

To understand precession, we need to appeal to the vectorial property of torque and angular momentum. The force of gravity, acting through the center of gravity of the top, creates a couple $\boldsymbol{G}$ about the point of contact, perpendicular to the plane of the spin and the vertical. Since there are no other couples acting, the angular momentum orthogonal to $\boldsymbol{G}$ is conserved. Thus, the vertical component of the spin angular momentum, which is orthogonal to $\boldsymbol{G}$, remains constant. The spin axis can therefore at most rotate about the vertical: this is precession.

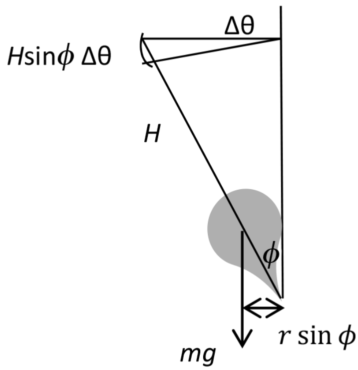


Figure 9.5: Precession

The rate of precession is given by equating the rate of change of angular momentum to the couple on the system. We give a purely vectorial derivation first, and then a slightly easier version from the geometry. The precession angular speed is a vector $\boldsymbol{\omega}_p$ in the direction of the unit vector $\mathbf{k}$ such that

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = \boldsymbol{\omega}_p \wedge \boldsymbol{H} = \omega_p \ \mathbf{k} \wedge \boldsymbol{H}. \tag{9.11}$$

The couple on the system is given by

$$\boldsymbol{G} = mgr\,\mathbf{k} \wedge \hat{\boldsymbol{H}}. \tag{9.12}$$

Equating (9.11) and (9.12) and using $\boldsymbol{H} = I\boldsymbol{\omega}$ gives

$$\omega_p = \frac{mgr}{I\omega}$$

for the rate of precession. Note that if ω decreases, for example, because of friction, the precession speeds up: this is verified by casual observation as a spinning top comes to rest, although this simple model does not tell us anything about the stability of the motion, which is in fact more complicated once $\omega_p > \omega$.

Alternatively, from figure,

$$\Delta H = H \sin\phi\,\Delta\theta = H \sin\phi\,\omega_p\,\Delta t.$$

Thus, the system precesses at a rate ω_p given by

$$\frac{dH}{dt} = H \sin\phi\,\omega_p = mgr\sin\phi,$$

from which, as before, with $H = I\omega$,

$$\omega_p = \frac{mgr}{I\omega}.$$

9.12 EULER EQUATIONS

As our final application of rigid body dynamics and the conservation of angular momentum, we answer the question: why do falling bodies tumble? By a tumbling motion, we mean the precession of the axis of spin in a falling body.

The precession of a free body is a bit more complicated than the systems we have dealt with so far. It arises if the body is not spherically symmetric. In the simplest case, the body will have cylindrical symmetry, with the moments of inertia about two perpendicular axes equal.

The equations of motion are obtained by considering the motion of the body in the rotating frame. If the time derivative in the rotating frame is denoted by d/dt, it is not true that $d\boldsymbol{H}/dt = \boldsymbol{G}$. This is because Newton's laws hold only in an inertial frame. If the frame of reference is rotating with angular velocity $\boldsymbol{\omega}$, then viewed from an inertial frame, there is an additional rate of change of the angular momentum of $\boldsymbol{\omega} \wedge \boldsymbol{H}$. Thus the correct equations of motion are

$$\left(\frac{d}{dt} + \boldsymbol{\omega}\wedge\right)\boldsymbol{H} = \boldsymbol{G}. \tag{9.13}$$

We now have to relate $\boldsymbol{H}$ to the moments of inertia. We shall not go into details: we just note that by definition of the principle moments of inertia (I_1, I_2, I_3) about the three principal axes of symmetry of the body, we have

$$\boldsymbol{H} = (I_1\omega_1, I_2\omega_2, I_3\omega_3).$$

In the case in question, the body is in free fall so $\boldsymbol{G} = 0$. Equation (9.13) in components becomes

$$I_1\dot{\omega}_1 + (I_2 - I_3)\omega_2\omega_3 = G_1,$$
$$I_2\dot{\omega}_2 + (I_3 - I_1)\omega_3\omega_1 = G_2,$$
$$I_3\dot{\omega}_3 + (I_1 - I_2)\omega_1\omega_2 = G_3.$$

In the case in question, the body is in free fall so $\boldsymbol{G} = 0$. If $I_2 = I_3 = C$ and we set $I_1 = A$, with $A > C$, we get

$$A\dot{\omega}_1 = 0,$$

so ω_1 = constant. Then

$$C\dot{\omega}_2 + (C - A)\omega_3\omega_1 = 0$$
$$C\dot{\omega}_3 + (A - C)\omega_1\omega_2 = 0$$

from which

$$C\omega_2\dot{\omega}_2 + C\omega_3\dot{\omega}_3 = 0,$$

and hence,

$$\omega_2^2 + \omega_3^2 = \text{constant}.$$

Thus, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ rotates about the one axis, that is, the body precesses at a rate determined by $A - C$. We can now see why a freely falling body tumbles.

9.13 CHAPTER SUMMARY

- The kinetic energy of a body rotating with angular speed ω is $E_K = \frac{1}{2}I\omega^2$, where $I = \sum \delta m r^2$ is the moment of inertia of the body about the axis of rotation

- The parallel axis theorem gives the moment of inertia I of a body of mass M about an axis parallel to one though the CM at a distance r: $I = I_{\mathrm{CM}} + Mr^2$
- The angular momentum of a body rotating with angular velocity $\boldsymbol{\omega}$ is $\boldsymbol{H} = I\boldsymbol{\omega}$
- The equation of motion of a body subject to a couple $\boldsymbol{G}$ about a fixed axis is $\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = \boldsymbol{G}$.
- Euler's equations for the general motion of a body are

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \boldsymbol{\omega} \wedge\right)\boldsymbol{H} = \boldsymbol{G}.$$

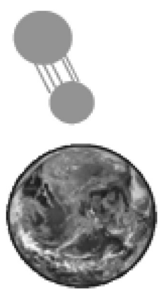
9.14 EXERCISES

1. A Fairground rotor consists of a cylindrical room which can be spun about its axis of symmetry. Intrepid members of the public stand with their backs to the wall while the room is spun up, at which point, the floor is removed. The people inside find themselves stuck to the wall. How fast must a 4 m rotor be spinning before the floor can be lowered?
2. Volvo engineers have experimented with a 0.2 m flywheel with a mass of 6 kg for regenerative breaking. The carbon fiber flywheel can rotate at up to 60 000 rpm. How much energy can it store?
3. A cylindrical body of radius a and mass M is released from rest on a plane inclined at angle θ. What is the condition that it rolls before slipping?
4. Precession of the Earth: Because it is not a perfect sphere the moment of inertia of the Earth about a polar axis, A, is greater than that about a perpendicular axis through the center, C. Show that the period of free precession is $\frac{2\pi C}{(A - C)\omega_1}$. Evaluate this and decide if it accounts for the precession of the equinoxes on a 26 000 year timescale.

CHAPTER 10

STABILITY OF MOTION

Problem: In what orientation does a dumbbell spaceship orbit?



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10.1 PERTURBATIONS

A dynamical system in equilibrium may be subject to an external perturbation. If the perturbation is small, we can compute its effect by approximation. There are two possible outcomes: the system returns to its equilibrium or it moves (or stays) away from it. In practice, in the latter case, the system may reach a new equilibrium or move chaotically: we cannot tell what eventually happens from the approximation for a small displacement if the displacement becomes large; all we can say is that the original equilibrium was unstable. The problem of determining the orientation of a spacecraft therefore reduces to finding the orientation that is stable to a small perturbation.

We can also use this approach as a trick to obtain solutions for problems where a part of the force can be regarded as a small addition, even if this has no physical meaning, or even if it is not in practice an actual perturbation. We shall begin by illustrating this with some examples.

10.2 CUBIC POTENTIAL

Consider a particle of mass m moving in the potential (sometimes called the anharmonic potential)

$$\phi(x) = \frac{1}{2}\omega^2 x^2 - \frac{1}{3}\varepsilon\omega^2 x^3, \tag{10.1}$$

where ε is a small quantity and the factor of 1/3 has been chosen for later convenience. The total energy per unit mass is therefore

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 - \frac{1}{3}\varepsilon\omega^2 x^3.$$

Differentiating with respect to t gives the equation of motion

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \ddot{x} + \omega^2 x - \varepsilon\omega^2 x^2 = 0. \tag{10.2}$$

Before we solve this, let us see what we might expect. The stationary points of the potential (10.1) are at

$$0 = \frac{\mathrm{d}\phi}{\mathrm{d}x} = \omega^2 x - \varepsilon\omega^2 x^2,$$

that is, at $x = 0$ and $x = 1/\varepsilon$. Since $\phi(x)$ is a cubic, it has one maximum and one minimum, and since $\phi \to -\infty$ as $x \to +\infty$, the stationary point at $x = 1/\varepsilon$ is the minimum. The particle therefore now oscillates about $x = 1/\varepsilon$.

In addition, the x^2 term has the same sign when x is both positive and negative; it represents a force in the positive x-direction.

(We can see this in several ways: $F = -\frac{\mathrm{d}\phi}{\mathrm{d}x} = +\varepsilon\,\omega^2 x^2$; or when x is positive, the x^3 term in the potential has the opposite sign from the simple harmonic term.) So the anharmonic term adds to the acceleration on the positive side and reduces the acceleration on the negative side. The particle therefore spends more time on the positive side of the origin, as we might expect from the previous argument that the center of oscillation has shifted to positive x.

Let us derive these results formally. As a first approximation to the solution of (10.2), we ignore the term in ε. Thus, from (10.2),

$$\ddot{x} = -\,\omega^2 x,$$

and hence, our first approximation is

$$x_0 = a\cos(\omega t + \phi). \tag{10.3}$$

As the next approximation, we could put $x = x_0 + \delta$, substitute this into (10.2) and ignore terms quadratic and above in the small quantities ε and δ. A quicker way is to approximate the small term we have so far, neglected using the approximate solution we have just obtained in (10.3). This gives

$$\ddot{x} + \omega^2 x \approx \varepsilon\omega^2 x_0^2 = \varepsilon\omega^2 a^2 \cos^2\left(\omega t + \phi\right).$$

We cannot solve this (directly) with a $\cos^2$ term on the right, so we use the double angle formula to turn this into a cosine term:

$$\ddot{x} + \omega^2 x = \frac{1}{2}\varepsilon\omega^2 a^2[1 + \cos 2(\omega t + \phi)].$$

The solution is

$$x_1 = a\cos\left(\omega t + \phi\right) + \frac{1}{2}\varepsilon\omega^2 a^2 - \frac{1}{2}\varepsilon a^2 \cos 2\left(\omega t + \phi\right).$$

The effect of the anharmonic term has been to shift the mean displacement to

$$\overline{x_1} = \frac{1}{2}\varepsilon\omega^2 a^2,$$

since the means of the cosine terms over a period are zero. So the particle does indeed spend more time with x positive. Applied to a pair of atoms in a solid, this means that the atoms spend more time at greater separation from their equilibrium than closer together. Since the amplitude of oscillation, a, depends on temperature, this is responsible for the expansion of solids on heating.

10.3 MOTION OF THE PLANET MERCURY

As a second example, we consider the motion of a planet according to general relativity. The effect of relativity is to add a small correction to the Newtonian equations of planetary motion (Section 7.19):

$$u'' + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2$$

for a planet with angular momentum per unit mass h in orbit about a star of mass M (with $u'' = \mathrm{d}^2u/\mathrm{d}\phi^2$). The small parameter here is $\varepsilon = h^2/c^2$, where c is the speed of light. So we can write

$$u'' + u = k + 3\varepsilon ku^2, \tag{10.4}$$

where $k = GM/h^2$. The first approximation (ignoring the term in ε) is

$$u_0 = k(1 + e\cos\phi).$$

If we look at what we expect to happen before we plunge into the solution, it will make the process of solving the equation easier. Equation (10.4) is similar to (10.1) except for the constant k on the right hand side. So we try to remove that by letting $u = v + k$:

$$v'' + \left(1 - 6k^2\varepsilon\right)v = 3\varepsilon k^3 + 3\varepsilon kv^2.$$

If we ignore the v^2 term now we have the equation of an ellipse, but with a modified period in ϕ, namely $2\pi/(1-6k^2\varepsilon)^{\frac{1}{2}}$. We see therefore that the perturbation increases the period of the orbit in space, which means that the body moves through a greater angle in returning to the same orbital distance $1/u$. If we focus on the point in the orbit at its closest to the parent star (the periastron), this precesses round the orbit.

With this in mind as a second approximation, we try

$$u_1 = (k + \delta_1)\left[1 + e\cos\left(\phi(1+\delta)\right)\right],$$

where we have assumed a shift in the periastron. Substituting back in (10.4) gives

$$-(k+\delta_1)(1+\delta)^2 e\cos(\phi(1+\delta)) + (k+\delta_1)[1+e\cos(\phi(1+\delta))$$

$$= k + 3\varepsilon k(k+\delta_1)^2[1+e\cos(\phi(1+\delta))]^2.$$

We now ignore terms that are of higher order than linear in the small quantities δ, δ_1, and ε to obtain

$$-2\delta ke\cos\phi + k + \delta_1 = k + 3\varepsilon k^3 + 6\varepsilon k^3 e\cos\phi.$$

Thus, comparing terms independent of ϕ and in $\cos\phi$, we get

$$\delta_1 = 3\varepsilon k^3 \quad \text{and} \quad \delta = -3\varepsilon k^2.$$

So the solution is

$$u = k\left(1 + 3\varepsilon k^2\right)(1 + e\cos\left(\left(1 - 3\varepsilon k^2\right)\phi\right).$$

In one orbit, u returns to its starting value in an angle

$$\triangle\phi = \frac{2\pi}{1 - 3\varepsilon k^2} \approx 2\pi\left(1 + 3\varepsilon k^2\right).$$

The periastron therefore advances by $6\pi\varepsilon k^2$ per orbit or by $6\pi\varepsilon G^2M^2/h^2c^2$ in physical units. For planets in orbit around the Sun, we speak of the precession of the perihelion. The predication of the correct perihelion precession of Mercury (which is the most easily measured in the solar system because of its short period and large eccentricity) is one of the observational tests of general relativity.

10.4 STABILITY: GENERAL FORMULATION

We now turn to the problem of stability to a small perturbation. It is often easier to treat each case independently, but we begin with a general formulation. Suppose we have an equation of motion

$$\frac{d^2x}{dt^2} = f(x,t),$$

with solution $x = x_0(t)$. To investigate the stability of this solution, we look at what happens if we add a small perturbation: $x = x_0(t) + \varepsilon(t)$:

$$\frac{d^2}{dt^2}(x_0 + \varepsilon) = f(x_0) + \varepsilon\left(\frac{\partial f}{\partial x}\right)_{x_0} + \ldots$$

by Taylor expansion, so

$$\frac{d^2\varepsilon}{dt^2} = \frac{\partial f(x_0(t),t)}{\partial x}\varepsilon.$$

If $\partial f/\partial x > 0$ at $x = x_0$, ε will grow exponentially; if $\partial f/\partial x < 0$ at $x = x_0$, ε will oscillate. Thus, the equilibrium point is stable if $\partial f/\partial x < 0$ at this point.

10.5 AN EXAMPLE OF STABILITY: NON-NEWTONIAN ORBITS

Consider a body moving in a plane orbit subject to an inverse nth power law $F = kr^{-n}$. The orbit equation is

$$u'' + u = ku^{n-2}, \tag{10.5}$$

which can be derived in exactly the same way as the inverse square case (Section 7.19). Consider a circular orbit $u = u_0 =$ constant $= ku_0^{n-2}$, or

$$ku_0^{n-3} = 1 \tag{10.6}$$

and suppose we add a perturbation $\varepsilon(\phi)$ so that $u = u_0 + \varepsilon$. Then from (10.5) to first order in ε

$$\varepsilon'' + \varepsilon = k(u_0 + \varepsilon)^{n-2} - u_0 = k(n-2)u_0^{n-3}\varepsilon = (n-2)\varepsilon,$$

where the zeroth-order terms cancel using (10.6). Therefore,

$$\varepsilon'' = (n-3)\varepsilon. \tag{10.7}$$

If $n < 3$, Equation (10.7) has oscillatory solutions so ε always remains small. The original solution before the perturbation is stable. If $n > 3$, the solution to (10.7) is a growing exponential, so the solution is unstable. For $n = 3$, the solution also grows (linearly). We therefore have stability if $n < 3$, and instability for larger n.

It is intriguing to consider how Newtonian gravity might appear if the dimension of space were >3. In three dimensions, the inverse square behavior of the gravitational force arises from the solution of Poisson's equation (Section 7.24) for the gravitational potential. If the gravitational potential in a space with higher dimensions satisfies the higher dimensional analog of Poisson equation, then the gravitational force will fall off faster than an inverse square. This means that circular orbits would not be stable in universes with spatial dimension >3.

For comparison, the general theory of Section (10.4) applied to Equation (10.5) asks us to evaluate

$$\frac{\partial f(u_0)}{\partial u} = \frac{\mathrm{d}}{\mathrm{d}u}\left(-u + ku^{n-2}\right)\Big|_{u=u_0} = -1 + (n-2)ku_0^{n-3} = n-3,$$

from which we obtain the conditions on stability as before.

10.6 A WARNING

As a warning against blindly applying the small perturbation algorithm, consider the equation of motion of a harmonic oscillator perturbed by a small quartic potential. The equation of motion is

$$\ddot{x} + \omega^2 x + \varepsilon x^3 = 0.$$

So as a first approximation (ignoring the term in ε), we try

$$x_0 = \sin \omega t.$$

Then, for our second approximation, we must solve

$$\begin{aligned}\ddot{x} + \omega^2 x &= -\varepsilon \sin^3 \omega t \\ &= -\frac{\varepsilon}{4} \mathcal{J}\left(3e^{i\omega t} - e^{3i\omega t}\right), \qquad (10.8)\end{aligned}$$

where $\mathcal{J}$ indicates the imaginary part and we have used the identity $\sin 3\theta = 3 \sin\theta - 4 \sin^3\theta$. We try

$$x_1 = At\, e^{i\omega t} + B e^{3i\omega t},$$

where the extra factor of t in the first term arises because $e^{i\omega t}$ satisfies the homogeneous equation. Substitution in (10.8) fixes A and B:

$$x_1 = \sin \omega t - \frac{\varepsilon}{32\omega^2} \sin 3\omega t + \frac{3\varepsilon}{8\omega} t \sin \omega t.$$

There is clearly a problem with the final term: it is supposed to be a small correction, but if we wait long enough, it will cease to be relatively small. The problem arises because we have tried to fix the period of oscillation to be unchanged as a result of the perturbation. In fact, the true approximation is

$$x = \sin\left(\omega + \frac{3\varepsilon}{8\omega}\right)t - \frac{\varepsilon}{32\omega^2} \sin 3\left(\omega - \frac{3\varepsilon}{8\omega}\right)t,$$

which we can find by following the procedure of Section (10.3).

10.7 SOLUTION TO PROBLEM

We are now in a position to address the problem of the orientation of the spacecraft.

To do this, we cannot treat the body in orbit as a point mass. So we need to make some sort of model (or approximation) to the shape of the body. Clearly, we want to keep it simple and not try to represent the detailed structure of the hypothetical space station. Figure 10.1 shows two possible models: we can make the dumbbells equal in mass and neglect mass of the joining section, or we can neglect the mass of the dumbbells and consider the mass to reside in the linear section. It does not matter which model we use to understand the problem and to illustrate the principles involved. We shall restrict ourselves to the first model in the following discussion:

Figure 10.1: Models of a space station. On the left, we consider two massive spheres connected by a massless strut; on the right, a massive strut connecting two massless spheres (which can therefore be neglected to give a cylindrical mass)

So we can return now to the problem of the orientation of the spacecraft. Figure 10.2 shows two possible orientations. The symmetry in each case means that there can be no net force or torque on the spacecraft, so in both of these pictures, the spacecraft is in equilibrium. The problem therefore is to determine which of these equilibria is stable.

So far, we have been looking at the spacecraft from the point of view of an orbit about a stationary Earth. That is to say, we have been using the Earth as our frame of reference. This is quite straightforward if the satellite can be described as a point mass, but for the motion of an extended body, it is quite complicated. Putting ourselves in the frame of reference of the

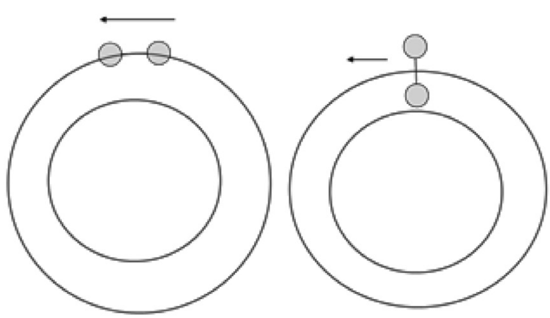

Figure 10.2: Two possible equilibrium orientations of the space station

satellite appears to remove the orbital motion, so the behavior of the satellite can be described more simply. However, we must make the transformation to the rotating frame of the satellite correctly. In a rotating frame of reference, Newton's laws of motion no longer hold. We say that a rotating frame is not an inertial frame: in a rotating frame bodies apparently subject to no forces appear to fly outward! We therefore have to amend Newton's laws by introducing additional forces to account for this behavior. An example, sufficient for our purpose here, is the centrifugal force. With the inclusion of centrifugal forces Newton's second law is valid in a rotating reference frame (see Chapter 8 for a general discussion).

Figure 10.3 shows why it is easier to consider the motion of the spacecraft in the rotating frame. Viewed from the Earth in an inertial frame the spacecraft rotates relative to a fixed direction as a result of its orbital motion. In the rotating frame, the equilibrium orientation remains fixed. We can therefore consider the motion about an equilibrium orientation most easily in the rotating frame provided that we include the outward centrifugal force.

Imagine then that the satellite is at rest and the Earth is spinning beneath it. To answer the question of stability, we imagine a small displacement of the satellite from equilibrium and look at the forces that then act on it.

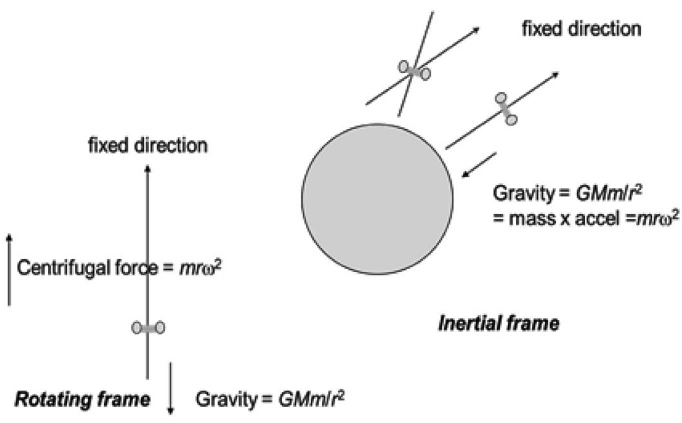

Figure 10.3: Orbits from the point of view of a rotating frame of reference

In case *I* on the left in Figure 10.4, dumbbell *a* is closer to the Earth than dumbbell *b*. What does this mean for the gravitational force on each? Since the strength of gravity falls off with distance from the Earth, the force on the closer dumbbell *a* is greater. We can see that this creates a moment about the CM that pulls the system back to the vertical. What about the outward centrifugal force? That on the dumbbell *a* is weaker than on *b*, which again acts to restore the system to equilibrium. So the equilibrium here is stable. A similar argument shows that the orientation II in Figure 10.4 is unstable.

The result depends on the nonuniformity of the gravitational field – the fact that it changes with distance – which generates an imbalance of gravitational forces. Recall from Chapter 7 that forces that arise from the nonuniformity of gravity are also called tidal forces.

Before we leave the subject of stability of equilibrium, we shall look at a slightly different approach from the point of view of energy. This is useful because it provides a picture that translates into many other contexts. It is also often easier to use for calculations. That is because, in contrast to forces, energy is a scalar quantity, so we do not have to worry about directions.

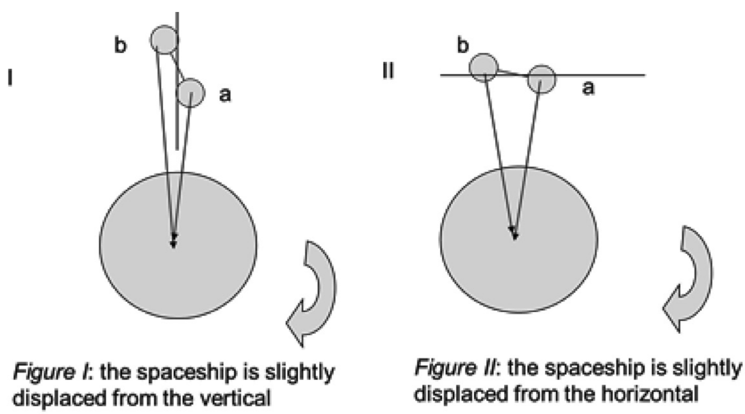

Figure 10.4: In a rotating frame, the spaceship is at rest while the Earth spins about a fixed center beneath it. As seen from the North pole, the Earth rotates anticlockwise in the rest frame of the satellite

In the rotating frame, the dumbbell is subject not only to the gravitational force but also to the centrifugal force. Both have an associated potential energy. For gravity, we know that the potential energy is *GMm/r*. For the energy associated with the centrifugal term, we must find the work done by the force $mr\omega^2$ in moving from the CM of the dumbbell at R to the center of the upper dumbbell at $R + \delta$ (Figure 10.5). This is the integral of the force over the distance:

$$\text{Centrifugal potential energy} = -\int mr\omega^2 \mathrm{d}r = -\frac{1}{2}mr^2\omega^2.$$

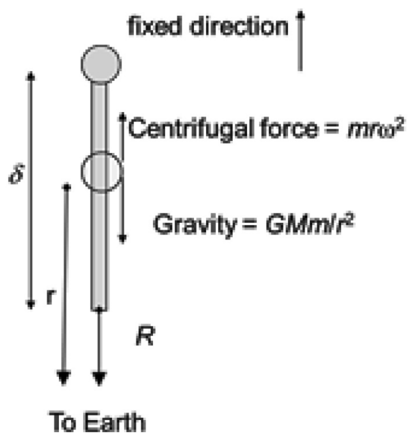

Figure 10.5: The centrifugal force and gravity in the frame of the spaceship

This is the loss in potential energy, since we are moving in the direction of the force, which does the work. So the potential energy is negative. The difference in potential energy between the CM and the dumbbell center is given by

$$-\frac{GMm}{R+\delta} - \frac{1}{2}m\omega^2(R+\delta)^2$$
$$-\left(-\frac{GMm}{R} - \frac{1}{2}m\omega^2 R^2\right).$$

Expanding this to lowest order in δ, we get

$$\Delta U = -\frac{GMm}{R}\left(1 - \frac{\delta}{R} + \frac{\delta^2}{R^2} + \dots\right) - \frac{1}{2}m\omega^2\left(2R\delta + \delta^2\right) + \frac{GmM}{R}$$

$$= -\frac{3}{2}m\omega^2\delta^2.$$

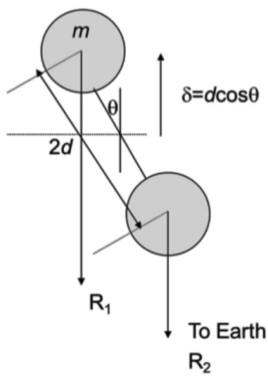

Figure 10.6: The dumbbell spacecraft is displaced by a small angle θ from its equilibrium

For the lower dumbbell, we get a similar contribution (it depends on δ^2 so has the same sign) and therefore the total potential energy of the two dumbbells is

$$\Delta U = -3m\omega^2\delta^2.$$

Imagine that the satellite is perturbed from its equilibrium position by a small angle θ (Figure 10.6). We can see from the diagram that the radial distance of each dumbbell from the center is $\delta = d\cos\theta$. So we now have an expression for energy as a function of the system coordinate, θ. Figure 10.7 shows a sketch of $\cos^2\theta$ plotted against θ. We can see that there is a minimum at $\theta = 0$ and the maximum at $\theta = \pi/2$. Thus orientation (a) is stable and (b) is unstable.

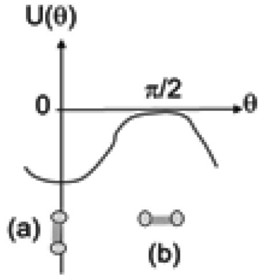

Figure 10.7: $\theta = 0$ is at the bottom of the potential energy curve, hence a stable point. $\theta = 0$ is at a maximum so is an unstable configuration

Finally, we can look at the equations of motion. The couple on the system is $-\partial\Delta U/\partial\theta$ and the kinetic energy of rotation about the CM is $\frac{1}{2}I\dot{\theta}^2$, so the equation of motion is

$$I\ddot{\theta} = -3m\omega^2 d^2\theta.$$

This is SHM with period $2\pi/\sqrt{\left(3m\omega^2 d^2\right)}$, so upon a small perturbation, the system oscillates about the equilibrium at $\theta = 0$.

10.8 PHASE PORTRAITS: HARMONIC OSCILLATOR

Often we want look not only at linear stability but also at what happens to an unstable system or at the range of stable behaviors open to a system. If we cannot solve the equations of motion (or even if we can), the problem can be approached by looking at the *phase plane*. This is a plot of the momentum (or speed) against position. The resulting plots are called *phase portraits*. We illustrate this first for the SHO

The equation of motion is

$$\ddot{x} = -\omega^2 x.$$

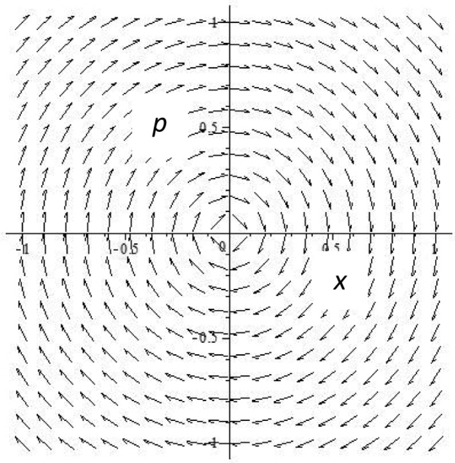

Figure 10.8: Phase portrait for a harmonic oscillator. The arrows show the trajectories of the oscillator in the phase plane (The plot was obtained using Maple symbolic computing software.)

We begin by rewriting this in dimensionless variables by defining $\tau = \omega t$:

$$\ddot{x} = -x. \tag{10.9}$$

To obtain the phase portraits, we need to rewrite this as a set of first-order equations. We define

$$\dot{x} = p, \tag{10.10}$$

so (10.9) becomes

$$\dot{p} = -x. \tag{10.11}$$

Finally, we eliminate time altogether by dividing (10.11) by (10.10):

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{x}{p}.$$

We can now graph the solutions in the x–p plane (Figure 10.8). If x and p are positive, the slope of the trajectory is negative, so an arrow in this quadrant slopes downward; similarly in the other quadrants.

Joining the arrows gives the trajectories through any given starting point. The plot in Figure (10.8) (and those below) was obtained using *Maple* symbolic computing software.

The advantage of this approach is that it shows us *all* solutions. In this case, the solutions are cycles (circles in fact). The disadvantage of this picture is that we lose any information about time.

10.9 PHASE PORTRAITS: DAMPED OSCILLATOR

If we add a linear damping term to the oscillator, the equations of motion become

$$\dot{x} = p,$$

$$\dot{p} = -x - rp,$$

where r is the damping rate constant. Figure 10.9 shows the system tending to a limit point at the origin.

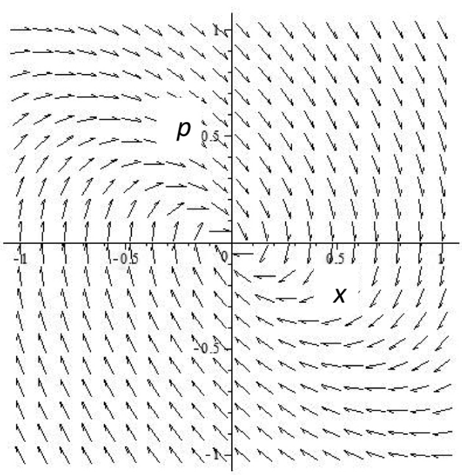

Figure 10.9: Phase portraits for the damped harmonic oscillator showing the systems tending to a limit point at the origin

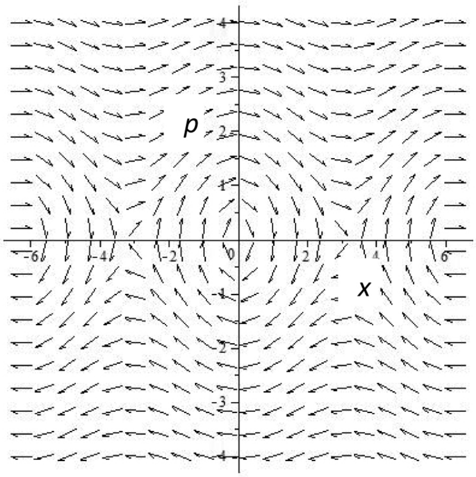

Figure 10.10: Phase portraits for a large amplitude pendulum; the coordinates $x = \pm\pi$ represent the same point

A finite amplitude pendulum has equation of motion:

$$\dot{x} = p,$$

$$\dot{p} = -\sin x,$$

where x is here the angular displacement from equilibrium. The behavior in Figure 10.10 is periodic since we identify $x = -\pi$ with $x = \pi$. For large enough displacements or momenta, the momentum does not change sign: on these trajectories, the pendulum swings over the top.

10.10 CHAOS

If we now add force and resistance to the finite amplitude pendulum, something strange happens. For certain values of the forcing amplitude, the motion becomes chaotic. This is difficult to see in the phase plane so we use a *Poincaré section*. In a Poincaré section, we plot the position of the point in the phase plane once every period of the driving force. Thus, a periodic motion is represented in a Poincare section as a single point. If we plot the representative point for the pendulum once, any transients have died out against the strength of the forcing term, for a given resistance and forcing frequency, we obtain the famous *bifurcation diagram* (Figure 10.11).

Note that for each value of the forcing strength, the plot shows the Poincare section. Initially, the motion is periodic so there is only a single point for a given value of the forcing. As the control is increased, the period doubles so the point returns to its initial location only after two cycles of the driving term. After a sequence of such period doublings, the motion becomes chaotic. The sequence repeats in compressed fashion infinitely often.

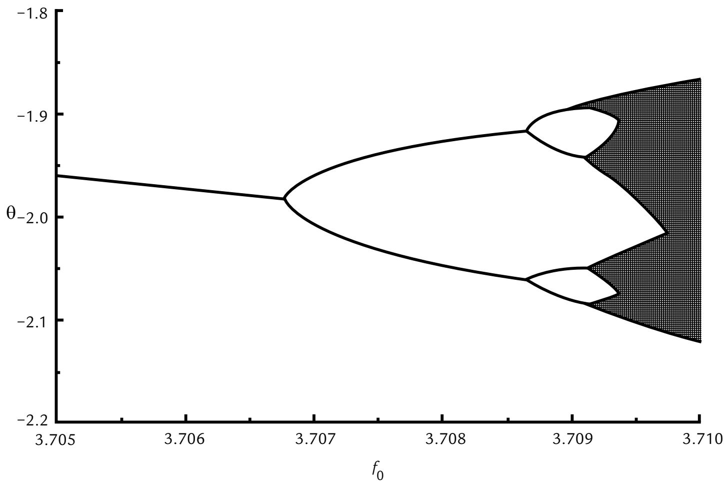

Figure 10.11: Bifurcation diagram for the damped driven pendulum. At each value of the forcing strength, the value of the steady state angle of the pendulum, θ at a fixed phase of the driving term is plotted (from *http://www.physics.udel.edu/~jim/PHYS460_660_11S/oscillations&chaos/The chaotic pendulum.htm*)

10.11 CHAPTER SUMMARY

- Equations of motion with a small parameter can be approximately solved in a power series in the small parameter

- If the small parameter represents a perturbation to an equilibrium solution, the solution can be used to test for stability of the equilibrium
- A phase portrait is a plot of the system in the coordinate–momentum plane.

10.12 EXERCISES

1. Construct a general theory of linear stability for an equation of the form

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right)$$

2. For potentials of the form

   (a) $U(x) = x(K - x)$ and (b) $U(x) = x(K - x)(H - x)$,

   where H and K are constants find the stable and unstable points of equilibrium.

3. The potential

$$\phi(x) = \omega^2 x^2 + \lambda x^4,$$

   where ω^2 and λ are constants, is often used to model departures from SHM. Investigate the equilibrium points and their stability for $\lambda > 0$ and $\lambda < 0$.

CHAPTER 11

LAGRANGIAN AND HAMILTONIAN MECHANICS

Problem: What is the general solution of the equations of motion of a dynamical particle system?

To understand the question, let us ask a simpler one: what is the general solution of a problem in statics? The answer comes from the *principle of virtual work*. Consider all the possible small displacements of the system. These are called virtual displacements; virtual work is the work done in making these displacements. The principle of virtual work then states that the equilibrium configurations of a mechanical system are those for which the virtual work is zero. Equivalently, a conservative system is in static equilibrium if the potential energy $U(x_i)$ is a minimum.

The equivalence of the two statements for a conservative system comes from

$$\delta U = \sum_i \frac{\partial U}{\partial x_i}\delta x_i = -\sum_i F_i \delta x_i = -\delta W,$$

so a minimum of the potential energy ($\delta U = 0$) corresponds to zero net work done ($\delta W = 0$) by the forces $F_i = -\partial U/\partial x_i$.

Can we state a similar principle for dynamical systems such that the trajectory of the system is obtained by a minimum (or extremum) principle?

11.1 PRINCIPLE OF LEAST ACTION

To keep things simple, at first, we consider particles with masses m_i at positions $x_i(t)$ moving in one dimension, with potential energy $U(x_i)$. The potential energy might arise from mutual interactions between the particles (e.g., their mutual gravity) or from an external source (e.g., the Earth's gravity). We then start from Newton's laws in the form:

$$m_i \ddot{x}_i = -\frac{\partial U}{\partial x_i}. \tag{11.1}$$

Assume now that the trajectory of the ith particle is varied by an amount $\delta x_i(t)$. Multiplying (11.1) through by δx_i, summing over i and integrating with respect to t, we get

$$\int_{t_0}^{t_1} \sum_i \left(m_i \ddot{x}_i \, \delta x_i + \frac{\partial U}{\partial x_i} \delta x_i \right) \mathrm{d}t = 0,$$

where the integral runs from an initial time t_0 to the current time t_1 and the sum is over all the particles in the system. We now integrate the first term by parts to get

$$\left[\sum m_i \dot{x}_i \delta x_i \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \sum_i \left(m_i \dot{x}_i \delta \dot{x}_i - \frac{\partial U}{\partial x_i} \delta x_i \right) \mathrm{d}t = 0. \tag{11.2}$$

Since the variation in the trajectories is a matter of choice, we can require all the $\delta x_i(t)$ to vanish at times t_0 and t_1. The first term on the left of (11.2) is then zero. Using

$$\delta \dot{x}^2 = 2\dot{x}\delta \dot{x},$$

and

$$\delta U = \sum_i \frac{\partial U}{\partial x_i} \delta x_i$$

allows us to put (11.2) in the form

$$-\delta \int \left(\sum \frac{1}{2} m_i \dot{x}_i^2 - U \right) \mathrm{d}t = 0,$$

where the variation δ refers to a variation of the path of the dynamical system, $x_i(t) \to x_i(t) + \delta x_i(t)$, with fixed end points, $\delta x_i(t_0) = \delta x_i(t_1) = 0$. Thus, we postulate that the quantity

$$S = \int_{t_0}^{t_1} \left(\sum \frac{1}{2} m_i \dot{x}_i^2 - U \right) \mathrm{d}t$$

is stationary along a trajectory of a dynamical system. The quantity S is called the *action* and the quantity $\mathcal{L} = E_\mathrm{K} - U$ (the kinetic energy *minus* the potential energy) is called a *Lagrangian* (for Newtonian dynamics) after Lagrange who synthesized these discoveries. Dynamical models often take the postulate of a form for the Lagrangian as their starting point.

The principle formalizes our heuristic energy minimization to obtain the equations of motion in Chapter 5. The important consequence of deriving the equations of motion from an action principle is that the equations of motion are then automatically consistent among themselves. This is why all modern dynamical theories, for example, in particle physics, start by postulating a form for the action.

11.2 EULER–LAGRANGE EQUATIONS

Rather than have to resort to first principles every time, we approach a new problem, we derive the form of the equations of motion for a general Lagrangian once and for all. Let the Lagrangian $\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i)$ be a function of the generalized coordinates q_i and $\dot{q}_i$, and let

$$S = \int_{t_0}^{t_1} \mathcal{L}(q_i, \dot{q}_i) \mathrm{d}t,$$

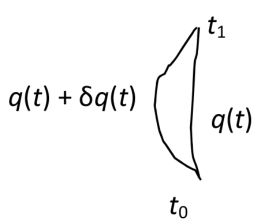

Figure 11.1: A trajectory $q(t)$ and a neighboring trajectory with fixed end points

where the integral is along a trajectory of the system $q_i = q_i(t)$. Note that we have used the symbols q_i here for the coordinates, and not x_i, to emphasize that the coordinates in the Lagrangian need not be Cartesian.

We imagine that we make a small change in the trajectory keeping the end points fixed (Figure 11.1). Then

$$S+\delta S=\int_{t_0}^{t_1}\mathcal{L}\left(q_i(t)+\delta q_i(t),\dot{q}_i(t)+\delta\dot{q}_i(t)\right)\mathrm{d}t,$$

$$\approx\int_{t_0}^{t_1}\left[\mathcal{L}\left(q_i(t),\dot{q}_i(t)\right)+\sum_i\left(\frac{\partial\mathcal{L}}{\partial q_i}\delta q_i(t)+\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta\dot{q}_i(t)\right)\right]\mathrm{d}t,$$

where we have expanded $\mathcal{L}$ to first order in the small variation (as a Taylor series). We integrate the final term by parts in order to get the integrand in terms of δq_i alone (and not δq_i):

$$\int_{t_0}^{t_1}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta\dot{q}_i(t)=\left[\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta q_i\right]_{t_0}^{t_1}-\int_{t_0}^{t_1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)\delta q_i\ \mathrm{d}t. \tag{11.3}$$

We have chosen $\delta q_i(t)$ such that it vanishes at the end points. So the first term of the right of (11.3) is zero. This gives us

$$\delta S=\int_{t_0}^{t_1}\left[\frac{\partial\mathcal{L}}{\partial q_i}\delta q_i(t)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)\delta q_i(t)\right]\mathrm{d}t,$$

which must hold for every choice of $\delta q_i(t)$. This can be the case only if the coefficient of $\delta q_i(t)$ vanishes. So we obtain finally

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}-\frac{\partial\mathcal{L}}{\partial q_i}=0 \tag{11.4}$$

for each coordinate i. The equations of motion in the form (11.4) are called the *Euler–Lagrange* equations.

It may seem that we have not gained much by writing Newton's laws in this elaborate fashion, but in fact, it is much easier to construct the scalar energies (kinetic and potential energies) for a complicated system than it is to apply the laws of motion from the forces directly. Let us look at some examples.

11.3 NEWTON'S LAWS

Given that our "derivation" in Section 11.1 was far from rigorous, we should check that the Euler–Lagrange equations really are equivalent to Newton's second law. For a collection of point masses m_i at x_i with potential energy $U(x_i)$, we have

$$\mathcal{L} = \sum_i \left(\frac{1}{2} m_i \dot{x}_i^2 - U(x_i) \right).$$

The generalized coordinates q_i are here the Cartesian coordinates x_i so the Euler–Lagrange equations become

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = \frac{\mathrm{d}}{\mathrm{d}t}(m_i \dot{x}_i) + \frac{\partial U}{\partial x_i} = 0,$$

or

$$m_i \ddot{x}_i = -\frac{\partial U}{\partial x_i} = F_i,$$

which is Newton's law, as required. Note that in differentiating with respect to $\dot{x}_i$, we treat it as an independent variable – think of it as v_i if you prefer. Note also that the sum in (11.5) is over $E_\mathrm{K} - E_\mathrm{P}$ not over the particle energies $E_\mathrm{K} + E_\mathrm{P}$.

11.4 SIMPLE HARMONIC OSCILLATOR

For an SHO, the Lagrangian is

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2,$$

and the Euler–Lagrange equation (with $q_i \equiv x$) is

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} \equiv \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = m\ddot{x} + m\omega^2 x = 0,$$

from which the usual oscillator equation follows. (This is of course how we knew to construct the Lagrangian; alternatively, we see it has the general form $E_K - E_P$ specialized to a harmonic oscillator.)

11.5 ACCELERATION IN POLAR COORDINATES

Let us do something a bit more useful. In Chapter 7, we saw how to derive the components of acceleration for motion in the plane in polar coordinates. The Euler–Lagrange equations give a much simpler derivation. For a particle of unit mass with kinetic energy only the Lagrangian is

$$\mathcal{L} = E_K = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right).$$

There are two Euler–Lagrange equations, one each for r and for θ. These give

$$a_r = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = \ddot{r} - r\dot{\theta}^2$$

and

$$a_\theta = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = r^2\ddot{\theta} + 2\dot{r}\dot{\theta},$$

which are the required components of acceleration.

11.6 ROTATING COORDINATE SYSTEM

The Lagrange method also gives a relatively easy way of obtaining the equations of motion in a rotating coordinate system. Consider a particle of unit mass at location $\boldsymbol{r}$. If it has velocity $\boldsymbol{v}$ in the rotating frame and velocity $\boldsymbol{V}$ in the inertial frame, then

$$\boldsymbol{V} = \boldsymbol{v} + \boldsymbol{\omega} \wedge \boldsymbol{r}.$$

The Lagrangian is

$$\mathcal{L} = \frac{1}{2}V^2 = \frac{1}{2}\boldsymbol{V}\cdot\boldsymbol{V} = \frac{1}{2}\left(\boldsymbol{v}\cdot\boldsymbol{v} + 2\boldsymbol{v}\cdot\boldsymbol{\omega}\wedge\boldsymbol{r} + \left[(\boldsymbol{\omega}\wedge\boldsymbol{r})\wedge\boldsymbol{\omega}\right]\cdot\boldsymbol{r}\right)$$

$$= \frac{1}{2}\left(\boldsymbol{v}\cdot\boldsymbol{v} + 2\boldsymbol{v}\wedge\boldsymbol{\omega}\cdot\boldsymbol{r} + \left[(\boldsymbol{\omega}\wedge\boldsymbol{r})\wedge\boldsymbol{\omega}\right]\cdot\boldsymbol{r}\right),$$

where we have exploited the cyclic properties of the scalar triple product.

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\boldsymbol{v}} - \frac{\partial\mathcal{L}}{\partial\boldsymbol{r}} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v} + \boldsymbol{\omega}\wedge\boldsymbol{r}) - \boldsymbol{v}\wedge\boldsymbol{\omega} - (\boldsymbol{\omega}\wedge\boldsymbol{r})\wedge\boldsymbol{\omega},$$

and, assuming $\boldsymbol{\omega}$ is constant,

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + 2\boldsymbol{\omega}\wedge\boldsymbol{v} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\boldsymbol{r}), \tag{11.5}$$

with additional terms in $\dot{\boldsymbol{\omega}}$ if $\boldsymbol{\omega}$ is not constant.

If you are troubled by the differentiation with respect to vectors, then we can add indices, so $\boldsymbol{v}$ is replaced by v_i, $\boldsymbol{r}$ by r_i, and $\boldsymbol{\omega}\wedge\boldsymbol{r}$ by $(\boldsymbol{\omega}\wedge\boldsymbol{r})_i$. Then

$$\mathcal{L} = \sum_i \frac{1}{2}\left(v_i v_i + 2v_i(\omega\wedge r)_i + r_i r_i \omega^2 - (\omega_i r_i)^2\right)$$

$$= \sum_i \frac{1}{2}\left(v_i v_i - 2r_i(\omega\wedge v)_i + r_i r_i \omega^2 - (\omega_i r_i)^2\right).$$

Then, from the first form for $\mathcal{L}$,

$$\frac{\partial\mathcal{L}}{\partial v_i} = v_i + (\boldsymbol{\omega}\wedge\boldsymbol{r})_i,$$

and from the second,

$$\frac{\partial\mathcal{L}}{\partial r_i} = -(\boldsymbol{\omega}\wedge\boldsymbol{v})_i + r_i\omega^2 - (\boldsymbol{\omega}.\boldsymbol{r})r_i = -(\boldsymbol{\omega}\wedge\boldsymbol{v})_i - (\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\boldsymbol{r}))_i.$$

Putting these together in the Euler–Lagrange equations gives the ith component of (11.5):

$$a_i = \frac{\mathrm{d}}{\mathrm{d}t}v_i + 2(\omega\wedge v)_i + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\boldsymbol{r})_i.$$

11.7 BEAD ON A WIRE

Here is a more interesting, if artificial, example that illustrates the power of the method for dynamical problems. Consider a circular hoop of wire rotating with constant speed in a horizontal plane about a point of its circumference with a bead of unit mass free to slide on the wire. What is the equation of motion of the bead relative to the wire? (Figure 11.2)

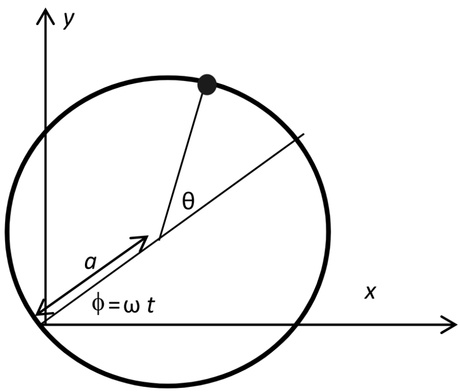

Figure 11.2: A bead is free to slide on a wire frame which is rotating in a horizontal plane about the origin

The position of the bead is given by

$$x = a\cos\omega t + a\cos(\omega t + \theta),$$

$$y = a\sin\omega t + a\sin(\omega t + \theta),$$

where θ, the generalized coordinate, is a function of t. The kinetic energy is

$$\mathcal{L} = E_{\mathrm{K}} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right)$$

$$= a^2\omega^2 + a^2(\dot{\theta} + \omega)^2 + 2a^2\omega\,(\dot{\theta} + \omega)\cos\theta,$$

which is the complete Lagrangian, because there is no potential energy. The equation of motion of the bead is

$$\ddot{\theta} + \omega^2\sin\theta = 0. \tag{11.6}$$

The interest of the problem is that the motion turns out to be the same as that of a compound pendulum under gravity.

Having obtained the result, we should be able to explain it. In fact, from the point of view of an observer rotating with the hoop, the bead slides on a wire in a gravitational field $a\omega^2$. In this frame of reference, we can derive (11.6) directly.

11.8 CYCLOIDAL PENDULUM

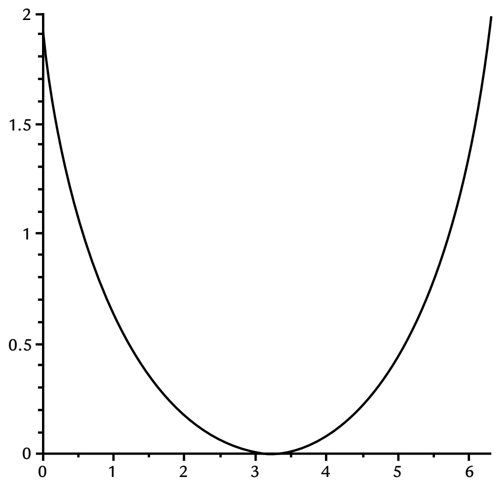

Figure 11.3: The cycloid $x = (\theta - \sin\theta)$, $y = (1 + \cos\theta)$

A bead slides without friction under gravity on a wire in the shape of a vertical cycloid

$$x = a(\theta - \sin\theta), \quad y = a(1 + \cos\theta)$$

where y measures the height of a point on the cycloid and $\theta\,(0 \le \theta \le 2\pi)$ is a generalized coordinate.

We want to find the equations of motion for θ. We have

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$$

$$= \frac{1}{2}ma^2(\dot{\theta} - \cos\theta\,\dot{\theta})^2 + \frac{1}{2}ma^2\sin^2\theta\,\dot{\theta}^2 - mga(1 + \cos\theta).$$

The Euler–Lagrange equations are therefore

$$\sin\frac{\theta}{2}\ddot{\theta}+\frac{1}{2}\dot{\theta}^2\cos\frac{\theta}{2}-\frac{g}{2a}\cos\frac{\theta}{2}=0.$$

If we put $u=\cos\frac{\theta}{2}$, we get

$$\ddot{u}+\frac{g}{4a}u=0,$$

which shows that the bead executes SHM. Note that we have not made the small angle approximation here: the cycloidal pendulum is exactly harmonic independent of the amplitude of the motion of the bead.

11.9 SPHERICAL PENDULUM

The Lagrange method is also useful if we want to look at small amplitude oscillations of a system. If we are interested only in small oscillations, then we can approximate the Lagrangian directly, rather than working out the exact equations of motion and then approximating. We shall use the spherical pendulum as an illustration. This comprises a bob of mass M on the end of a weightless rod free to move on the interior surface of a sphere.

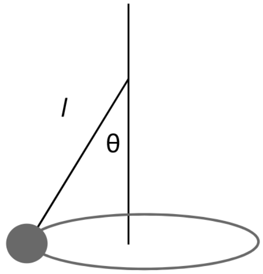

Figure 11.4: The spherical pendulum

Let θ be the angle of the rod with the vertical, ϕ the rotation angle, and let the length of the rod be l. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}Ml^2\dot{\phi}^2\sin^2\theta + \frac{1}{2}Ml^2\dot{\theta}^2 - Mgl(1 - \cos\theta).$$

The first step is to remove all the constants as far as possible. An additive constant can be neglected. An overall constant in the Lagrangian will not contribute to the Euler–Lagrange equations so we can take out a constant factor. One might be tempted to divide through by Ml^2, but this would leave a dimensional factor in the potential energy. Instead, we take out a factor of Mgl;

$$\mathcal{L} = \frac{1}{2}\frac{l}{g}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\sin^2\theta + \frac{1}{2}\frac{l}{g}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \cos\theta.$$

We can also rescale the time t in favor of a dimensionless time τ

$$\tau = \left(\frac{g}{l}\right)^{\frac{1}{2}} t.$$

This gives us a new Lagrangian with much less clutter:

$$\mathcal{L} = \frac{1}{2}\phi'^2\sin^2\theta + \frac{1}{2}\theta'^2 + \cos\theta,$$

where the prime denotes differentiation with respect to τ. There are two Euler–Lagrange equations, one for each of the generalized coordinates θ and ϕ. These are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathcal{L}}{\partial\phi'} - \frac{\partial\mathcal{L}}{\partial\phi} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\sin^2\theta\,\phi'\right) = 0,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathcal{L}}{\partial\theta'} - \frac{\partial\mathcal{L}}{\partial\theta} = \theta'' - \phi'^2\cos\theta\sin\theta + \sin\theta = 0.$$

One solution is

$$\phi' = p_\phi/\sin^2\theta_0 = \phi'_0 = \text{constant},$$

$$\theta = \theta_0 = \cos^{-1}\frac{1}{\phi_0'^2} = \text{constant},$$

where p_ϕ is a constant (proportional to the angular momentum per unit mass). As we might expect, the bob swings round at a constant angle to the vertical related to its speed. In other words, the bob behaves as a conical pendulum with period

$$T_\phi = \frac{2\pi}{\phi_0'}\sqrt{\frac{l}{g}}. \tag{11.7}$$

In general, we cannot solve the equations of motion analytically. In Figure 11.5, we show the phase portraits the $\theta' - \theta$ plane for the case $p_\phi = 2$ computed numerically. The pendulum oscillates in θ between an upper and lower value.

We now perturb the motion of the conical pendulum. Let

$$\theta = \theta_0 + \varepsilon,$$
$$\phi = \phi_0 + \delta.$$

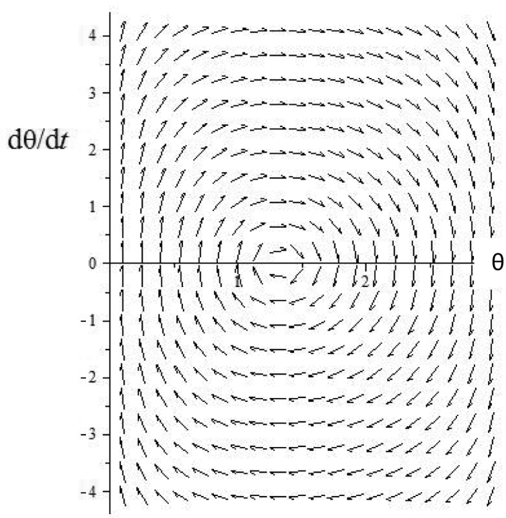

Figure 11.5: Numerical solutions of the θ – equation showing trajectories in the phase plane. θ oscillates between upper and lower values

In deriving the approximate Lagrangian, we can ignore constant terms, since these will not contribute to the Euler–Lagrange equations. We can also ignore terms linear in ε and δ because the action stationary at the equilibrium solution (so the first order variation vanishes). We therefore need to calculate the second-order terms in ε and δ. The Lagrangian becomes approximately:

$$\mathcal{L} = \frac{1}{2}(\delta'^2 \sin^2\theta_0 + 4\delta' \varepsilon \phi_0' \cos\theta_0 \sin\theta_0$$
$$+ \varepsilon^2(\cos^2\theta_0 - \sin^2\theta_0)\phi_0'^2 + \varepsilon'^2 + \varepsilon^2 \cos\theta_0).$$

The equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\delta'}\right) - \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\delta} = \frac{\mathrm{d}}{\mathrm{d}\tau}(\delta' \sin^2\theta_0 + 2\varepsilon\, \phi_0' \cos\theta_0 \sin\theta_0) = 0, \qquad (11.8)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\varepsilon'}\right) - \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\varepsilon} = \varepsilon'' - 2\delta'\phi_0' \cos\theta_0 \sin\theta_0$$
$$-\varepsilon\, \phi_0'^2(\cos^2\theta_0 - \sin^2\theta_0 + \cos\theta_0) = 0. \qquad (11.9)$$

Equation (11.8) gives

$$\delta' \sin^2\theta_0 + 2\varepsilon'\phi_0' \cos\theta_0 \sin\theta_0 = \text{constant}. \qquad (11.10)$$

If we consider the case where we do not perturb the angular momentum, the constant in (11.9) can be set to zero. We can then use (11.9) to eliminate δ' from (11.9) that becomes

$$\varepsilon'' = -\varepsilon\, \phi_0'^2\, (1 + 2\cos^2\theta_0 - \cos\theta_0).$$

This is SHM: the pendulum oscillates in altitude θ with a period

$$T_\theta = \frac{2\pi}{\phi_0'(1 + 2\cos^2\theta_0 - \cos\theta_0)^{\frac{1}{2}}}\sqrt{\frac{l}{g}}.$$

The period of the azimuthal (ϕ) motion is also perturbed slightly from its original value T_ϕ (Equation (11.7)), but in general, the two periods will be unequal. The azimuthal angle at which the altitude θ is a maximum (or minimum) therefore precesses round the orbit.

11.10 COMPOUND PENDULUM

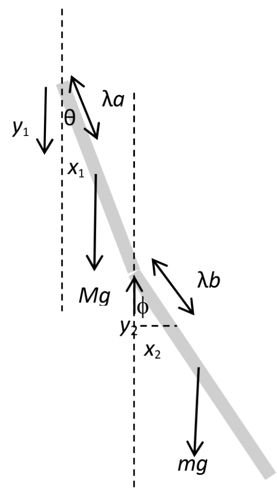

Figure 11.6: A compound pendulum

Many textbook examples of the Lagrange method are artificial because they are chosen to give equations of motion that can be solved analytically. In practice the Lagrange method lets us set up the equations of motion of real systems which can then be solved numerically. Here is an example where we use the method to obtain the equations of motion, and extract some information, but would need numerical methods to get a complete solution. The example is the compound pendulum. This comprises two rods of lengths $2a$ and $2b$ and mass M and m, respectively, which can pivot freely about a support at the top of one and about the point where they are joined. This is a model of a golf swing (pivoted at the shoulder and wrist) and of the medieval siege weapon, the trebuchet. In both cases, the idea is to put more energy into the projectile at the end of the pendulum (the golf ball or the missile) than would be possible with an unjointed pendulum.

The potential energy of the system (Figure 11.6) is straightforward:

$$\begin{aligned} U &= Mga(1-\cos\theta) + mg\,[2a(1-\cos\theta) + b(1-\cos\phi)] \\ &= (M+2m)\,ga(1-\cos\theta) + mgb(1-\cos\phi). \end{aligned}$$

The kinetic energy is more complicated. It comprises the rotational energy of the upper strut plus the rotational and translational energy of the lower strut. In these cases, it is often safer to deduce the kinetic energy from first principles. Taking the top pivot as origin, the x and y coordinates of a point on the upper strut a distance λa from the pivot are

$$x_1 = \lambda a \sin\theta, \qquad y_1 = -\lambda a \cos\theta,$$

where $0 \le \lambda \le 2$.

For the lower strut, the corresponding coordinates are

$$x_2 = 2a \sin\theta + \lambda b \sin\phi, \qquad y_2 = -2a\cos\theta - \lambda b \cos\phi.$$

Assuming the mass density, ρ per unit length, is uniform, the elements of mass are $\rho a \mathrm{d}\lambda$ on the top link and $\rho b \mathrm{d}\lambda$ on the bottom strut. The kinetic energy is

$$\frac{1}{2}\rho\int_0^2\Big[a\,\mathrm{d}\lambda\left(\lambda a\cos\theta\,\dot{\theta}\right)^2 + a\,\mathrm{d}\lambda\left(\lambda a\sin\theta\,\dot{\theta}\right)^2 + b\,\mathrm{d}\lambda\left(2a\cos\theta\,\dot{\theta} + \lambda b\cos\phi\,\dot{\phi}\right)^2 + b\,\mathrm{d}\lambda\left(2a\sin\theta\,\dot{\theta} + \lambda b\sin\phi\,\dot{\phi}\right)^2\Big].$$

Performing the integrations, we get

$$E_K = \left(\frac{2}{3}M + 2m\right)a^2\dot{\theta}^2 + \frac{2}{3}mb^2\,\dot{\phi}^2 + mab\,\dot{\theta}\,\dot{\phi}\cos(\phi - \theta),$$

where $M = 2\rho a$ is the mass of the upper strut and $m = 2\rho b$ is the mass of the lower strut.

To simplify slightly, we take the case where $a = b$ and to simplify drastically we assume $m \ll M$ and that we are interested only in small oscillations. Then, ignoring terms higher than quadratic,

$$\mathcal{L} = \left(\frac{2}{3}M + 2m\right)a^2\dot{\theta}^2 + \frac{2}{3}ma^2\dot{\phi}^2 + ma^2\dot{\theta}\,\dot{\phi} - \frac{1}{2}(M + 2m)ga\theta^2 - \frac{1}{2}mga\phi^2.$$

We can simplify this by taking out a common factor of Mga and transforming to a dimensionless time

$$\tau = \sqrt{\frac{g}{a}}t.$$

Then, letting $\mu = m/M$ be our small parameter,

$$\mathcal{L} = \left(\frac{2}{3} + 2\mu\right)\theta'^2 + \frac{2}{3}\mu\phi'^2 + \mu\theta'\phi' - \frac{1}{2}(1+2\mu)\theta^2 - \frac{1}{2}\mu\phi^2.$$

The Euler–Lagrange equations of motion, for θ and ϕ, respectively, are

$$4\left(\frac{1}{3} + \mu\right)\theta'' + \mu\phi'' + (1+2\mu)\theta = 0 \tag{11.11}$$

$$\frac{4}{3}\phi'' + \theta'' + \phi = 0.$$

We deduce that the two eigenfrequencies are

$$\omega_{\pm} = \left(\frac{\sqrt{3}}{2} \pm 0.67\sqrt{\frac{m}{M}}\right)\sqrt{\frac{g}{a}}. \tag{11.12}$$

To check, note that if $\mu = 0$, there is no ϕ-term in the Lagrangian, so the only equation of motion is (11.11) with μ set to 0; this is SHM with a (dimensionless) frequency of $\sqrt{3}/2$, which agrees with (11.12) in this limit.

Thus the frequency of the pendulum is split into two close frequencies by the addition of the small mass of the lower strut. We can refer to Section 8.10 to see what happens rather than repeating the exercise. The system beats at the difference of the two eigenfrequencies and energy is transferred between the two struts on the corresponding long timescale. From (11.11), we see that the amplitude of the θ oscillation is m/M times that of the ϕ-oscillation. This is reasonable if the energy is to be transferred between the larger and smaller masses. Thus, if we set the top strut in motion initially, what we see is the top strut oscillating with a small amplitude which gradually decreases as the lower strut starts to oscillate with a (relatively) large amplitude; the situation is then gradually reversed. We can guess from this how a trebuchet works. Under the right conditions, the energy of the massive arm is largely concentrated into the smaller arm and hence into the projectile. It is analogous to the whiplash effect.

11.11 SMALL OSCILLATIONS REVISITED

With a general formulation, we can approach the problem of small oscillations in greater generality than in Chapter 10. Rather than starting from the equations of motion and adding a small perturbation to the coordinates, we manipulate the Lagrangian to the form where it yields directly the small oscillations equations.

Let us start from a general form for the Lagangian:

$$\mathcal{L} = \sum_{ij} \frac{1}{2} T_{ij}\, \dot{q}_i \dot{q}_j - \sum_i V(q_i).$$

For small displacements about equilibrium, $q_i = q_{0i} + \varepsilon_i$,

$$\mathcal{L} = \mathcal{L}_0 + \sum_{ij} \left(\frac{1}{2} T_{ij} \dot{\varepsilon}_i \varepsilon_j - \frac{\partial^2}{\partial x_i \partial x_j} V(x_0) \varepsilon_i \varepsilon_j \right),$$

where the linear terms have disappeared because q_0 satisfies the equations of motion. Then, taking ε_i as the generalized coordinates of the system, the Euler–Lagrange equations give

$$\sum_j \left(T_{ij}\, \ddot{\varepsilon}_j + V_{ij} \varepsilon_j \right) = 0, \tag{11.13}$$

where we have defined

$$V_{ij} = \left[\frac{\partial^2 V}{\partial x_i \partial x_j} \right]_{\boldsymbol{x} = \boldsymbol{x}_0}.$$

Equation (11.13) is a linear equation with constant coefficient (because T_{ij} and V_{ij} are evaluated at the equilibrium point $\boldsymbol{x}_0$). So we look for a solution $\varepsilon_j = a_j e^{i\omega t}$, giving

$$\sum_j (-\omega^2 T_{ij} a_j + V_{ij} a_j) = 0.$$

This set of equations has only the trivial solution $a_j = 0$ for all j unless the system of equations is singular. Thus (in matrix notation), for a nontrivial solution to the oscillation amplitudes, we require

$$\det(\omega^2 T - V) = 0. \tag{11.14}$$

Solving (11.14) for ω gives the eigenfrequencies (or normal modes) of the system. The system is stable if the imaginary part of ω satisfies $\mathcal{I}(\omega) \leq 0$. If T is the identity matrix, then the squared eigenfrequencies, ω^2, are the eigenvalues of the matrix V.

11.12 AN EXAMPLE

Consider masses m_1 and m_2 attached at the trisection points of a massless spring. Let the masses oscillate in a line as in Figure 11.7. The problem is to ascertain whether we can adjust the masses m_1 and m_2 to obtain any given pair of eigenfrequencies.

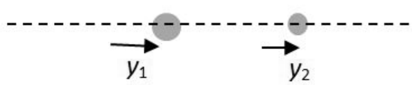

Figure 11.7: Mass points on a massless spring

Let the displacements of the masses from equilibrium be y_1 and y_2. The kinetic energy of the system is

$$E_K = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2,$$

and the potential energy is

$$U = \frac{1}{2} k y_1^2 + \frac{1}{2} k (y_1 - y_2)^2 + k y_2^2,$$

where k is the elastic constant of the spring. Then (in the notation of Section 11.11)

$$T = \begin{pmatrix} \frac{\partial^2 E_K}{\partial y_1 \partial y_1} & \frac{\partial^2 E_K}{\partial y_1 \partial y_2} \\ \frac{\partial^2 E_K}{\partial y_2 \partial y_1} & \frac{\partial^2 E_K}{\partial y_2 \partial y_2} \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix},$$

and

$$V = \begin{pmatrix} \frac{\partial^2 U}{\partial y_1 \partial y_1} & \frac{\partial^2 U}{\partial y_1 \partial y_2} \\ \frac{\partial^2 U}{\partial y_2 \partial y_1} & \frac{\partial^2 U}{\partial y_2 \partial y_2} \end{pmatrix} = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}.$$

The frequencies of the normal modes (eigenfrequencies) are given by

$$\det\begin{pmatrix} \omega^2 m_1 - 2k & -k \\ -k & \omega^2 m_2 - 2k \end{pmatrix} = 0. \tag{11.15}$$

Let $k/m_1 = \omega_1^2$ and $k/m_2 = \omega_2^2$. Then (11.15) becomes

$$\omega^4 - 2\left(\omega_1^2 + \omega_2^2\right)\omega^2 + 3\omega_1^2\omega_2^2 = 0.$$

The solutions for the frequencies of the normal modes are

$$\omega^2 = \left(\omega_1^2 + \omega_2^2\right) \pm \sqrt{\left(\omega_1^2 + \omega_2^2\right)^2 - 3\omega_1^2\omega_2^2}. \tag{11.16}$$

Now we choose any two frequencies ω_a and ω_b say and see if we can solve (11.16) for ω_1 and ω_2. There are a few shortcuts that are worth noting.

We have to solve

$$\omega_a^2 = \left(\omega_1^2 + \omega_2^2\right) + \sqrt{\left(\omega_1^2 + \omega_2^2\right)^2 - 3\omega_1^2\omega_2^2}, \tag{11.17}$$

$$\omega_b^2 = \left(\omega_1^2 + \omega_2^2\right) - \sqrt{\left(\omega_1^2 + \omega_2^2\right)^2 - 3\omega_1^2\omega_2^2}, \tag{11.18}$$

for ω_1 and ω_2. Adding (11.17) and (11.18) gives

$$\frac{1}{2}\left(\omega_a^2 + \omega_b^2\right) = \omega_1^2 + \omega_2^2. \tag{11.19}$$

Subtracting (11.17) from (11.18) and squaring gives

$$\frac{1}{4}\left(\omega_a^2 - \omega_b^2\right)^2 = \left(\omega_1^2 + \omega_2^2\right)^2 - 3\omega_1^2\omega_2^2. \tag{11.20}$$

Using (11.19) to eliminate $(\omega_1^2 + \omega_2^2)$ from (11.20) with some rearrangement gives

$$\omega_1^2\omega_2^2 = \frac{1}{3}\omega_a^2\omega_b^2. \tag{11.21}$$

We can now use (11.19) and (11.21) to find ω_1 and ω_2. We have

$$\left(\omega_1 - \omega_2\right)^2 = \omega_1^2 - 2\omega_1\omega_2 + \omega_2^2 = \frac{1}{2}\left(\omega_a^2 + \omega_b^2\right) - \frac{2}{\sqrt{3}}\omega_a\omega_b,$$

and

$$\left(\omega_1 + \omega_2\right)^2 = \omega_1^2 + 2\omega_1\omega_2 + \omega_2^2 = \frac{1}{2}\left(\omega_a^2 + \omega_b^2\right) + \frac{2}{\sqrt{3}}\omega_a\omega_b.$$

Extracting ω_1 and ω_2 is now straightforward. We obtain finally:

$$\omega_1 = \frac{1}{2}\left[\frac{1}{2}\left(\omega_a^2 + \omega_b^2\right) + \frac{2}{\sqrt{3}}\omega_a\omega_b\right]^{\frac{1}{2}} + \frac{1}{2}\left[\frac{1}{2}\left(\omega_a^2 + \omega_b^2\right) - \frac{2}{\sqrt{3}}\omega_a\omega_b\right]^{\frac{1}{2}},$$

$$\omega_2 = \frac{1}{2}\left[\frac{1}{2}\left(\omega_a^2 + \omega_b^2\right) + \frac{2}{\sqrt{3}}\omega_a\omega_b\right]^{\frac{1}{2}} - \frac{1}{2}\left[\frac{1}{2}\left(\omega_a^2 + \omega_b^2\right) - \frac{2}{\sqrt{3}}\omega_a\omega_b\right]^{\frac{1}{2}}.$$

The issue now is whether these values are real for all values of ω_a and ω_b. Let $x = \omega_a/\omega_b$. Then

$$\frac{1}{2}\left(\omega_a^2 + \omega_b^2\right) - \frac{2}{\sqrt{3}}\omega_a\omega_b = 0 \text{ if } x^2 - \frac{4}{\sqrt{3}}x + 1 = 0,$$

that is, $x = \sqrt{3}$ or $1/\sqrt{3}$. Between these values the argument of the square root is negative. Thus, we can adjust the masses to achieve frequency ratios in the range $\omega_a/\omega_b > \sqrt{3}$ or $\omega_a/\omega_b < 1/\sqrt{3}$. In other words, there is a bandgap of nonallowed frequencies.

11.13 HAMILTONIAN MECHANICS

The Lagrange equations provide us with a general method, but not with a general solution of the equations of motion. Hamilton tried to develop the theory further to achieve such a general solution. Although he did not succeed, his work today is of upmost importance for the analysis of dynamical systems and the development of quantum mechanics.

The idea of Hamiltonian mechanics is to write the equations of motion in first-order form. To do this, we first define a *canonical momentum*

$$p = \frac{\partial \mathcal{L}}{d\dot{q}}. \tag{11.22}$$

The *Hamiltonian* function, H, a function of q and p, is then obtained from

$$H = p\dot{q} - \mathcal{L}. \tag{11.23}$$

In the general case that the Lagrangian (and hence the Hamiltonian) are time dependent, we get for a small variation of coordinates

$$\begin{aligned}\delta H &= \delta p\dot{q} + p\delta\dot{q} - \frac{\partial \mathcal{L}}{\partial q}\delta q - \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta\dot{q} + \frac{\partial H}{\mathrm{d}t}\delta t\\ &= \delta p\dot{q} - \dot{p}\delta q + \frac{\partial H}{\partial t}\delta t,\end{aligned}$$

using (11.22) and the Euler–Lagrange equations. Here $\dot{q}$ and $\dot{p}$ are considered to be functions of p and q. Since variations δq, δp, and δt are independent, we obtain

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p}, \tag{11.24}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q}, \tag{11.25}$$

and

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}. \tag{11.26}$$

These are the *equations of motion in Hamiltonian form.*

The action is now

$$S = \int (p\dot{q} - H)\,dt \tag{11.27}$$

integrated along a trajectory. Minimizing S (with respect to p and q) we retrieve Hamilton's equations (11.25) and (11.24).

Next we show that the momenta can be obtained from the action:

$$p = \frac{\partial S}{\partial q}. \tag{11.28}$$

To do this, we vary the end point of the trajectory in the integral in (11.27). We have then

$$\delta S = \delta \int (p\,dq - H\ dt) = p\delta q - H\delta t$$

because this is what we add to the integral when we vary the final end point by δt and $\delta q(t)$. Since δq and δt are independent, we obtain both (11.28) and

$$\frac{\partial S}{\partial t} = -H(q, p). \tag{11.29}$$

This gives us an important result: the action S is the solution of the partial differential equation

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0. \tag{11.30}$$

Equation (11.30) is the *Hamilton–Jacobi equation*. A general solution of this equation would in principle be the solution to all mechanics problems. More practically, it gives us a novel picture of classical mechanics which provides a starting point for the quantum picture.

More generally, if a system is defined by more than one generalized coordinate, $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$, then the momenta are

$$\boldsymbol{p} = \nabla S, \tag{11.31}$$

and the Hamilton–Jacobi equation becomes

$$\frac{\partial S}{\partial t} + H\left(q_1, q_2, \ldots, q_n, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \ldots, \frac{\partial S}{\partial q_n}\right) = 0.$$

The Hamilton–Jacobi equation can be thought of as showing the evolution of the surfaces S = constant in space (in the q-coordinates). From (11.31), this means that the particle trajectories are orthogonal to the surfaces S = constant (because ∇S is the normal to S = constant).

11.14 CONSERVATION LAWS AND NOETHER'S THEOREM

Suppose that $\mathcal{L}$ is independent of position. Then the dynamics is unchanged by translation $q \to q + \varepsilon$. Also, from the Euler–Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0,$$

and hence

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \text{constant}.$$

So the conservation of momentum is linked to the invariance of a system under translation.

Similarly, if $\mathcal{L}$ is independent of an angle, ϕ say, then

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \text{constant}$$

expresses the conservation of the corresponding component of angular momentum.

Finally, if $\mathcal{L}$ is independent of t, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L}\right) = \ddot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} + \dot{q}\frac{\partial \mathcal{L}}{\partial q} - \frac{\partial \mathcal{L}}{\partial q}\dot{q} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q} = 0,$$

so

$$\dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} = \dot{q}p - \mathcal{L} = H = \text{constant} = E.$$

We shall show that the constant E is the energy, by looking at the special case

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 - U(q).$$

For this Lagrangian, we have

$$H = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} = \frac{1}{2}m\dot{q}^2 + U(q),$$

which is indeed the total energy. We also see that

$$H = E.$$

Thus, if the Lagrangian (and Hamiltonian) does not depend explicitly on time (except through the dynamical variables p and q), then the Hamiltonian is numerically equal to the energy, and the energy is conserved. So time translation invariance is related to conservation of energy.

These results are part of a general theorem (*Noether's theorem*) linking symmetries of the Lagrangian to conserved quantities. To state the theorem, we first have to distinguish between continuous and discrete symmetries. For example, reflection symmetry ($\boldsymbol{x} \to -\boldsymbol{x}$) is a discrete symmetry, while translation ($\boldsymbol{x} \to \boldsymbol{x} + \delta\boldsymbol{x}$) is a continuous symmetry, since the system can be varied by arbitrarily small amounts. Roughly speaking then, Noether's theorem states that: to a continuous symmetry of the action, there corresponds a quantity that does not change over time.

11.15 ENERGY AND THE HAMILTONIAN

In this section, lets add a few remarks about energy. First, we have just seen that, where it exists and is independent of time, the Hamiltonian gives us an expression for the conserved energy of a system. Thus, we do not expect to be able to describe a system in which there is dissipation (energy is not conserved) directly in terms of a Hamiltonian. (We would introduce an environment which can exchange energy with the system.) Where it exists, the Hamiltonian tells us how to construct the energy of each of the agents of the system and how these agents can exchange energy through their interactions. Note that energy is not itself an agent – it cannot "do" anything but is exchanged between agents when they act on each other.

Confusion can arise when we allow passive agents into the system, which is agents that can act but are not acted upon. In this case, the interaction energy appears as a potential energy in the subsystem under consideration. Here are two examples.

Consider a body in a constant gravitational field raised to a height z. The Hamiltonian is

$$H(z,p) = \frac{p^2}{2m} + mgz.$$

It might appear from this that there is some store of energy called "poptential energy" in the only active agent in the system. This leads to confusion in locating where this energy is and how it "acts." We can clarify this by including a fuller set of agents, namely the body and the gravitational field:

$$H(p,q,\phi) = \frac{p^2}{2m} + mz\nabla\phi\big|_z + \frac{1}{2}\int(\nabla\phi)^2\,\mathrm{d}V.$$

We see here there are two stores of energy (the kinetic energy of the body and the energy of the field – the first and last terms on the right) and an interaction that transfers energy between the two. (In Newtonian mechanics, the speed of gravity is infinite so the field energy is instantaneously shared with the Earth; if you prefer you can think of the more complete set of agents as the body and the

Earth.) The potential energy now appears as a sort of pathway for the transfer of energy between the interacting agents.

Let us look at another example: a mass (bob) on a spring. Here energy of the spring makes a somewhat mysterious appearance as the potential energy of the bob. A more complete description explains the mystery. The agents are the spring and the bob. The bob has a kinetic energy and the spring has a kinetic and elastic energy and they interact through the highly inelastic glue that binds them together. The Hamiltonian has the form

$$H\left(p, z, P, Z\right) = \frac{p^2}{2m} + \frac{P^2}{2M} + \frac{1}{2}kZ^2 + \Lambda\left(z - Z\right)^2,$$

where the lowercase symbols refer to the bob and the uppercase ones to the spring and the interaction $\Lambda(z - Z)^2$ ties the two together. If we let $M \to 0$ (massless spring) and $\Lambda \to +\infty$ (inelastic glue), we can recover the SHM equation for the bob. The point is that in the fuller description of the system, the energy stores are exchanged between the two agents not between two types of energy of the bob.

One final point. We are not arguing that we should abandon the highly useful concept of potential energy. The point is that if you find the idea of energy troublesome in some context, then a Hamiltonian formulation for the system as a whole will probably sort things out. It is not necessary to be able to write down explicit expressions for the Hamiltonian to outline this fuller description.

11.16 ACTION ANGLE VARIABLES AND INTEGRABLE SYSTEMS

An important distinction can be made in the types of behavior of dynamical; systems related to their integrability. We shall explain this through an example.

Consider the harmonic oscillator again. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2.$$

Note that H is a function of p and q. Now define new coordinates J and θ by

$$p = \sqrt{2m\omega J}\cos\theta,$$

$$q = \sqrt{\frac{2J}{m\omega}}\sin\theta.$$

The Hamiltonian becomes

$$H = \omega J.$$

The coordinates J and ω are known as *action angle variables*. In terms of these variables, the system is completely integrable: Hamilton's equations become

$$J = \text{constant}, \quad \theta = \omega t + \text{constant},$$

so the motion takes place on a circle. Whenever we can find a transformation that puts the Hamiltonian into action angle form, $H = J \cdot \boldsymbol{\omega}$, we say the system is integrable. The motion takes place on the n-dimensional generalization of a circle which is a torus in n-dimensions. (It is not possible to picture this beyond $n = 2$!)

However, not all systems are integrable. Suppose we give our oscillator a small perturbation:

$$H = J\omega + \varepsilon f(J, \omega)$$

Hamilton's equations become

$$\frac{dJ}{dt} = -\varepsilon\frac{\partial f}{\partial\theta},$$

$$\frac{d\theta}{dt} = \omega + \varepsilon\frac{\partial f}{\partial J}.$$

Next we expand f in a complex Fourier series:

$$f(J,\theta) = \sum_k f_k(J)e^{ik\theta}.$$

So

$$\frac{\mathrm{d}J}{\mathrm{d}t} = -\varepsilon \sum_k ik\mathrm{e}^{ik\theta} f_k(J),$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega + \varepsilon \sum_k f_k'(J)\mathrm{e}^{ik\theta}.$$

The solutions are, to first order in ε,

$$\theta = \omega t + \varepsilon \sum_k f_k'(J_0)\mathrm{e}^{ik\omega t},$$

$$J = \varepsilon \sum_k (ik\mathrm{e}^{ik\omega t}) \frac{f_k(J_0)}{k\omega}.$$

In n-dimensions, we should have

$$\boldsymbol{\theta} = \boldsymbol{\omega} t + \varepsilon \sum_{\boldsymbol{k}} f_{\boldsymbol{k}}'(J_0)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\omega} t},$$

$$\boldsymbol{J} = \varepsilon \sum_{\boldsymbol{k}} (i\boldsymbol{k}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\omega} t}) \frac{f_{\boldsymbol{k}}(\boldsymbol{J}_0)}{\boldsymbol{k}\cdot\boldsymbol{\omega}}.$$

So $J \to \pm\infty$ if there are modes with $\boldsymbol{k} \cdot \boldsymbol{\omega} = 0$. This is a signal of non-integrable behavior and leads to systems exhibiting chaotic (unpredictable) motions in the absence of dissipation. This behavior is called *Hamiltonian chaos*.

11.17 QUANTUM THEORY

Our final task is to prepare to leave classical mechanics for quantum mechanics. There are two routes. The first is the *Heisenberg picture* that we obtain as follows.

We write the Hamiltonian equations of motion in terms of a Poisson bracket, defined by

$$\{F, G\} = \left(\frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial G}{\partial q}\frac{\partial F}{\partial p}\right).$$

Then with $F = p$ and $G = q$

$$\{p,q\} = -1, \qquad \{p,p\} = 0, \qquad \{q,q\} = 0,$$

and

$$\{q,H\} = \frac{\partial H}{\partial p} = \dot{q},$$

$$\{p,H\} = -\frac{\partial H}{\partial q} = \dot{p}.$$

We get the quantum equations of motion by replacing q by the operator $\hat{q}$ and p by the operator $\hat{p}$ and the Poisson bracket by a commutator

$$[\hat{p},\hat{q}] = \hat{p}\,\hat{q} - \hat{q}\,\hat{p}.$$

The alternative approach to quantum mechanics is the *Schrödinger picture*. Here we start from the Hamilton–Jacobi equation with Cartesian coordinate $q = x$:

$$E = H\left(\frac{\partial S}{\partial x}, x\right).$$

We replace H by the operator

$$H\left(-i\hbar\frac{\partial}{\partial x}, x\right),$$

and S by the wave function ψ. So, for example, if

$$H = \frac{p^2}{2m} + V(x),$$

we get

$$E\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi,$$

the (time independent) Schrödinger wave equation. Classical mechanics then turns out to be the geometrical optics approximation to the waves of the quantum theory.

11.18 CHAPTER SUMMARY

- The equations of motion of a conservative dynamical system can be obtained by minimizing the action of the system
- The action is the integral over time of the Lagrangian, $\mathcal{L}$ = Kinetic energy – Potential energy = $T - U$
- The Euler–Lagrange equations of motion for a system with Lagrangian $\mathcal{L}$ and generalized coordinates q_i are
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$
- The frequencies of small oscillations are given by $\det(\omega^2 T - V) = 0$, where the matrices T and V are defined by the quadratic terms in the kinetic and potential energies, respectively
- The equations of motion can be written in terms of the Hamiltonian, $H(q, p)$ in first-order form
$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p} \text{ and } \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q}.$$
- If the Hamiltonian is not explicitly dependent on time, then $H = E$, the total energy
- Noether's theorem states that to a continuous symmetry of the action there corresponds a quantity that does not change over time

11.19 EXERCISES

1. Figure 11.8 shows two rigid *massless* rods with masses M_1, M_2, and M_3 attached. The lower rod has length $2L$ and is supported on a frictionless pivot about its midpoint. The upper rod has length $2l$ and is linked to the top of the first rod by a frictionless pivot at its bottom end.

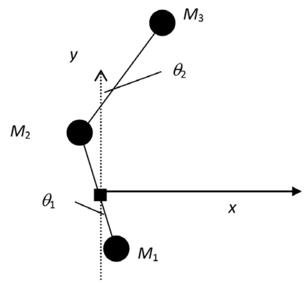

Figure 11.8: Exercise 1

Show that the coordinates (x, y) of the three masses relative to a coordinate system through the support are

$$x_1 = L\sin\theta_1,\ \ y_1 = -L\sin\theta_1,$$
$$x_2 = -L\sin\theta_1,\ \ y_2 = -L\cos\theta_2,$$
$$x_3 = -L\sin\theta_1 + 2l\sin\theta_2, y_3 = L\cos\theta_1 + 2l\cos\theta_2.$$

Hence (or otherwise) show that the Lagrangian for the motion of the rods is

$$\mathcal{L} = \frac{1}{2}M_1L^2\dot{\theta}_1^2 + \frac{1}{2}M_2L^2\dot{\theta}_2^2 + \frac{1}{2}M_3L^2\dot{\theta}_1^2 + 2M_3l^2\dot{\theta}_2^2$$
$$+ 2LM_3(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)\dot{\theta}_1\dot{\theta}_2$$
$$+ M_1gL\cos\theta_1 - M_2gL\cos\theta_1 - M_3g(L\cos\theta_1$$
$$+ 2l\cos\theta_2).$$

Derive the equations of motion for small oscillations about the position of equilibrium with the rods upright. By seeking a solution with θ_1 and $\theta_2 \propto \exp(i\omega t)$ (or otherwise) show that there is a mode of oscillation that is stable.

Why is this position nevertheless unstable?

2. In spherical polar coordinates, the kinetic energy of a unit mass particle is

$$E_K = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2.$$

Find the components of acceleration in spherical polar coordinates.

3. The Lagrangian for a particle in an electromagnetic field is

$$\mathcal{L} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \phi\left(x, y, z\right) + xA + yB + zC,$$

where A, B, and C are functions of position x, y, and z. Derive the equations of motion.

4. A triatomic molecule is modeled as a central mass M joined by springs, with spring constants k, to two masses m in a linear configuration. Find the modes of oscillation and their frequencies.

5. A uniform rod of mass M, length l, is freely pivoted at one end. Investigate its possible motion under gravity.

6. A point particle of mass m is attached to the end of a massless rod the other end of which is free to slide on a planar curve of the form $y = f(x)$. Investigate the motion of the system.

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